The Complexity of GARCH Option Pricing Models

YING-CHIE CHEN*, YUH-DAUH LYUU** AND KUO-WEI WEN***
*Department of Finance
Taiwan University
Taipei, 106 Taiwan
E-mail:r95723051@ntu.edu.tw
**Department of Computer Science and Information Engineering
Taiwan University
Taipei, 106 Taiwan
E-mail: lyuu@csie.ntu.edu.tw
***Department of Computer Science and Information Engineering
Taiwan University
Taipei, 106 Taiwan
E-mail: wenkuowei@gmail.com

When using trees to price options, the standard practice is to increase the number of partitions per day, n, to improve accuracy. But increasing n incurs computational overhead. In fact, raising n makes the popular Ritchken-Trevor tree under non-linear GARCH (NGARCH) grow exponentially when n exceeds a typically small threshold. Worse, when this happens, the tree cannot grow beyond a certain maturity because of the impossibility of finding valid probabilities. Lyuu and Wu prove the results under NGARCH. They also prove that, by making the tree track the mean value, valid probabilities can always be found if n does not exceed some threshold; furthermore, the growth rate of the tree's size is only quadratic in n. This paper completes that line of research by proving that LGARCH, AGARCH, GJR-GARCH, TS-GARCH and TGARCH share the same properties as NGARCH. The theoretical results are verified by numerical experiments.

Keywords: GARCH, path dependency, trinomial tree, option pricing

1. INTRODUCTION

It is well-known that the stationary lognormal process as a model for asset price dynamics fails to explain such phenomena as fat tails of return, volatility clustering and the negative correlation between stock returns and their volatilities. In order to address these issues, Bollerslev and Taylor independently propose the generalized autoregressive conditional heteroskedastic (GARCH) models [1, 11]. GARCH-type models have since proved successful for modeling financial time series. These models assume that the current variance of asset price is affected by the long-run average variance rate, past variances, and past squared returns. Since the prediciability of variance is important for options and GARCH-type models have been found useful to predict time-varying and persistent variance, GARCH option pricing models have drawn much attention [6, 8]. Duan shows how options can be priced when the volatility dynamics of the price of the underlying asset follows the GARCH process [3]. However, because of the massive path dependence inherent in GARCH models, pricing is complicated and naive trees incur

exponential complexity.

Ritchken and Trevor present the first practical tree algorithm (called the RT tree in the paper) for pricing options under GARCH models [9]. Later, Cakici and Topyan propose a slightly revised version (called the CT tree), which employs only realized variances instead of interpolated variances in building the GARCH tree [2]. For higher precision, it is a standard practice to pick a larger n (the number of partitions per day). Unfortunately, Lyuu and Wu prove that the sizes of both RT and CT trees grow exponentially in n when n exceeds some thresholds [7]. What is worse, when this happens, the trees cannot grow beyond a certain maturity date because valid probabilities are no longer possible for both trees. This maturity may be only a few days from now. Lyuu and Wu also introduce a new tree, the mean-tracking trinomial tree (called the MT tree in the paper) to address the issues [7]. They prove that MT solves the shortened-maturity problem and the tree size is only quadratic in n if n does not exceed some threshold. The MT tree is furthermore quite accurate in practice. We will drop the CT tree from now on as it yields the same conclusions as the RT tree.

Lyuu and Wu's results apply to the non-linear GARCH model (NGARCH) of Engle and Ng [4, 7]. This paper extends their investigations to a host of other GARCH models: LGARCH of Bollerslev and Taylor, AGARCH of Engle and Ng, GJR-GARCH of Glosten et al., TS-GARCH of Taylor and Schwert, TGARCH of Zakoian [1, 4, 5, 10-12]. This paper will present explosion and non-explosion thresholds for all the GARCH models above. Specifically, like NGARCH, this paper proves that (1) the RT trees for LGARCH, AGARCH, GJR-GARCH, TS-GARCH and TGARCH explode if n exceeds some threshold and (2) the MT trees for them grow only quadratically in n if n does not exceed some threshold. For the numerical accuracy of the MT trees, see Lyuu and Wu [7].

The paper proceeds as follows. Section 2 presents the GARCH models and introduces the change of measure that makes them risk-neutral in order to price options. Section 3 reviews the RT tree. Section 4 reviews the MT tree. Sections 5 to 10 derive thresholds for exponential explosion using the RT tree and thresholds for quadratic growth using the MT tree under NGARCH, LGARCH, AGARCH, GJR-GARCH, TS-GARCH, and TGARCH. Numerical results are presented in each section to corroborate the theoretical results. Section 11 concludes.

2. GARCH MODELS AND CHANGE OF MEASURE

Let S_t denote the asset price at time t and h_{t+1}^2 the conditional variance of the return over $[t, t+1]$. The asset price is assumed to be conditionally lognormally distributed under probability measure P,

$$\ln \frac{S_{t+1}}{S_t} = r + \lambda h_t - \frac{1}{2} h_t^2 + h_t \varepsilon_{t+1}, \tag{1}$$

where ε_{t+1} has mean zero and variance 1 given information at time t under measure P, r is the constant one-period risk-free rate of return and λ is the constant unit risk premium.

In order to develop a GARCH option pricing model, we have to change the measure

from P-measure to a risk-neutral Q-measure. According to Duan, the risk-neutral process (i.e., under measure Q) for the logarithmic price $y_t \equiv \ln S_t$ is

$$y_{t+1} = y_t + r - \frac{1}{2}h_t^2 + h_t\xi_{t+1}, \tag{2}$$

where ξ_{t+1} has mean zero and variance 1 given information at time t under measure Q and r is the same as the above [3].

We assume h_{t+1}^2 follows a GARCH process. Six GARCH models that the paper will investigate are described below. The variable ε_{t+1} is replaced by $\xi_{t+1} - \lambda$ in each model to make it risk-neutral.

NGARCH: Non-linear asymmetric GARCH model of Engle and Ng [4]

The P-measure process is

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 \left(\varepsilon_{t+1} - c\right)^2,$$

and the Q-measure process is

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 \left(\xi_{t+1} - \lambda - c\right)^2 \tag{3}$$

where $\beta_0 > 0$, $\beta_1 \geq 0$, and $\beta_2 \geq 0$. NGARCH generalizes the following LGARCH.

LGARCH: Linear GARCH model of Bollerslev [1] and Taylor [11]

The P-measure process is

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 \varepsilon_{t+1}^2,$$

and the Q-measure process is

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 \left(\xi_{t+1} - \lambda\right)^2, \tag{4}$$

where $\beta_0 > 0$, $\beta_1 \geq 0$, and $\beta_2 \geq 0$.

AGARCH: Asymmetric GARCH model of Engle and Ng [4]

The P-measure process is

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta_2 \left(\varepsilon_{t+1} h_t + c\right)^2,$$

and the Q-measure process is

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta_2 \left(\xi_{t+1} h_t - \lambda h_t + c\right)^2, \tag{5}$$

where $\beta_0 > 0$, $\beta_1 \geq 0$, and $\beta_2 \geq 0$.

GJR-GARCH: Glosten et al. [5]

The P-measure process is

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 \varepsilon_t^2 + \beta_3 h_t^2 \max\left(0, -\varepsilon_{t+1}\right)^2,$$

and the Q-measure process is

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 \left(\xi_{t+1} - \lambda\right)^2 + \beta_3 h_t^2 \max\left(0, -\left(\xi_{t+1} - \lambda\right)\right)^2, \tag{6}$$

where $\beta_0 > 0$, $\beta_1 \geq 0$, $\beta_2 \geq 0$ and $\beta_1 + \beta_2 \geq 0$.

TS-GARCH: Taylor [10] and Schwert [10]

The P-measure process is

$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t \left|\varepsilon_{t+1}\right|,$$

and the Q-measure process is

$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t \left|\xi_{t+1} - \lambda\right|, \tag{7}$$

where $\beta_0 > 0,\quad \beta_1 \geq 0,$ and $\beta_2 \geq 0$.

TGARCH: Threshold GARCH model of Zakoian [12]

The P-measure process is

$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t \left|\varepsilon_{t+1}\right| + \beta_3 h_t \max\left(0, -\varepsilon_{t+1}\right),$$

and the Q-measure process is

$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t \left|\xi_{t+1} - \lambda\right| + \beta_3 h_t \max\left(0, -\left(\xi_{t+1} - \lambda\right)\right), \tag{8}$$

where $\beta_0 > 0,\quad \beta_1 \geq 0,\quad \beta_2 \geq 0$ and $\beta_2 + \beta_3 \geq 0$.

3.THE RT TREE

This section describes the RT algorithm for NGARCH. Later, the underlying trinomial tree structure will be used to investigate the explosion thresholds for the other 5 GARCH models mentioned in Section 2.

In the RT tree, a grid is laid out first and the logarithmic prices y_t are positioned on the nodes of the grid. Each node contains bivariate states $\left(y_t, h_t^2\right)$. The distance between two adjacent logarithmic prices (nodes) equals

$$\gamma_n = \frac{\gamma}{\sqrt{n}}, \tag{9}$$

where $\gamma = h_0$ and each day is partitioned into *n* periods. As a result, every logarithmic price y_t must be some integer multiple of γ_n. The logarithmic price in the next period is approximated by a discrete random variable that takes 3 values. The probabilities for the state $\left(y_t, h_t^2\right)$ to follow the up, middle, and down branches are denoted by p_u, p_m and p_d, respectively. Note that there are $2n+1$ nodes after one day.

The jump size is the magnitude by which the state $\left(y_t, h_t^2\right)$'s 3 successor prices are spaced. It must be some integer multiple η of γ_n and measures how wide the tree fans out. The number η is called the jump parameter. We will pick the jump size later. After one day, the RT spans over $2n\eta+1$ nodes. See Fig. 1 for illustration. Observe that the mean and variance of y_{t+1} given $\left(y_t, h_t^2\right)$ at day t are

$$\mathrm{E}_t\left[y_{t+1}\right] = y_t + r - \frac{1}{2}h_t^2,$$
$$\mathrm{Var}_t\left[y_{t+1}\right] = h_t^2.$$

The mean and the variance of the discrete random variable have to match the mean and the variance of the continuous random variable, respectively. Furthermore, the probabilities for the up, middle, and down branches must sum to 1:

$$p_u + p_m + p_d = 1.$$

From the above three equations, the 3 probabilities can be obtained:

$$p_u = \frac{h_t^2}{2\eta^2\gamma^2} + \frac{r - \left(h_t^2/2\right)}{2\sqrt{n}\eta\gamma}, \tag{10}$$

$$p_m = 1 - \frac{h_t^2}{\eta^2\gamma^2}, \tag{11}$$

$$p_d = \frac{h_t^2}{2\eta^2\gamma^2} - \frac{r - \left(h_t^2/2\right)}{2\sqrt{n}\eta\gamma}. \tag{12}$$

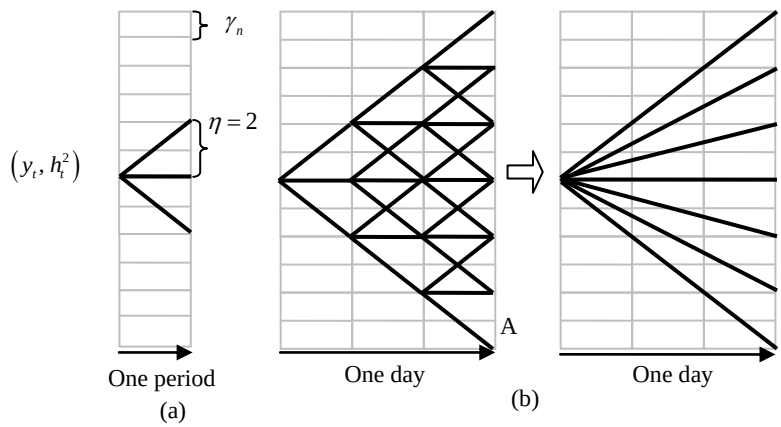


Fig. 1. The distance between two adjacent logarithmic prices γ_n, the jump parameter η and the $(2n+1)$-nomial tree. (a) The logarithmic prices have to be on the nodes. In this example, the jump parameter η is 2 because adjacent branches are separated by 2 grids. (b) With $n = 3$, there will be 7 nodes after one day, resulting in a 7-nomial tree after getting rid of the intermediate nodes.

We then pick the jump parameter to make sure the probabilities for the up, middle and down branches are valid. As p_m must lie in between 0 and 1, from equation (11),

$$\eta \geq \frac{h_t}{\gamma}, \tag{13}$$

which implies the minimum jump parameter must increase with the size of h_t^2, which is the variance of y_{t+1}. The jump parameter η is found by going through

$$\left\lceil \frac{h_t}{\gamma} \right\rceil, \left\lceil \frac{h_t}{\gamma} \right\rceil + 1, \left\lceil \frac{h_t}{\gamma} \right\rceil + 2, \ldots$$

until valid p_u, p_m and p_d that lie between 0 and 1 are found or until their non-existence can be confirmed. Recall that the number of nodes that follows y_t at date t is $2n+1$ in the next day. When $n = 1$, we have a trinomial tree, and when $n = 3$, it

becomes a 7-nomial tree after intraday nodes are removed as shown in the right-hand side of Fig. 1(b).

Technically, state $\left(y_t, h_t^2\right)$ at date t is followed by $\left(y_{t+1}, h_{t+1}^2\right)$ at date $t+1$, where

$$\begin{aligned} y_{t+1} &= y_t + j\eta\gamma_n, \\ h_{t+1}^2 &= \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 \left(\xi_{t+1} - c - \lambda\right)^2, \\ \xi_{t+1} &= \frac{j\eta\gamma_n - \left(r - \frac{h_t^2}{2}\right)}{h_t}, \\ j &= 0, \pm 1, \pm 2, \ldots, \pm n. \end{aligned} \tag{14}$$

For example, node A in Fig. 1(b) represents state $\left(y_{t+1}, h_{t+1}^2\right)$, where

$$y_{t+1} = y_t - 3 \cdot 2\gamma_n,$$

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 \left(\frac{-3 \cdot 2\gamma_n - \left(r - \frac{h_t^2}{2}\right)}{h_t} - c - \lambda \right)^2.$$

The probabilities for the $2n+1$ branches of the $(2n+1)$-nomial tree in Fig. 1(b) are given by

$$\text{Prob}\left(y_{t+1} = y_t + j\eta\gamma_n\right) = \sum_{j_u, j_m, j_d} \binom{n}{j_u \;\; j_m \;\; j_d} p_u^{j_u} p_m^{j_m} p_d^{j_d},$$

where $j_u, j_m, j_d \geq 0$, $n = j_u + j_m + j_d$ and $j = j_u - j_d$. Lyuu and Wu give a simple way to calculate those probabilities [7].

We use Fig. 2 to explain the RT tree-building algorithm. Let node A represent the node at day 0 with logarithmic price y_0 and variance h_0^2. At day 1, there is only one path to each of nodes B, C and D; so there is also only one variance, h_1^2, found by formula (14) for these nodes. Let us move on to day 2. At node E, there are two paths to it, so there are two variances, each of which picks its own jump parameter η. In general, the number of possible values of h_t^2 at a node equals the number of paths to that node. As the tree grows larger, the number of variances at a node may grow exponentially. To suppress it, the RT algorithm records the maximum $h_{\max}^2$ and the minimum $h_{\min}^2$ of all the variances at a node. Each node on the RT tree keeps a total of only K variances. So in addition to $h_{\max}^2$ and $h_{\min}^2$, there are $K-2$ other variances linearly interpolated between $h_{\max}^2$ and $h_{\min}^2$; these $K-2$ variances are

$$h_{\min}^2 + j\frac{h_{\max}^2 - h_{\min}^2}{K-1}, \quad j = 0, 1, 2, \ldots, K-1.$$

We mention that other interpolation schemes such as the loglinear scheme have also been suggested. Interpolation schemes may affect the numerical accuracy but not the computational complexity, which is our main concern. Hence this paper will not investigate this issue.

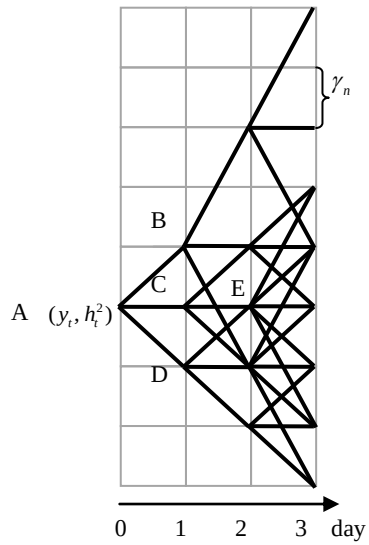


Fig. 2. An RT tree over three days with $n=1$. Node B has jump parameter 2 and node E has jump parameters 1 and 2, one for each of the two paths to it.

4.THE MEAN-TRACKING (MT) TREE

Now we introduce the MT algorithm for NGARCH proposed by Lyuu and Wu as an alternative to RT [7]. It will be used to derive the non-explosion thresholds for the other 5 GARCH models later.

Unlike RT, MT lets the middle branch of the tree track the mean of y_t as closely as possible. Define the mean of the next logarithmic price y_{t+1} minus the current logarithmic price y_t by

$$\mu \equiv E\left[y_{t+1}\right] - y_t = r - \frac{h_t^2}{2}. \tag{15}$$

Since $E\left[y_{t+1}\right] = y_t + \mu$ is most likely not on a node of the grid, MT chooses a point A with logarithmic price $y_t + \alpha\gamma_n$ that is closest to $y_t + \mu$; in other words,

$$\left|a\gamma_n - \mu\right| \le \frac{\gamma_n}{2} \tag{16}$$

(see Fig. 3).

Like the RT tree, the MT tree is a $(2n+1)$-nomial tree if one day is partitioned into n periods. After one day, MT spans over

$$2n\eta + 1 \tag{17}$$

nodes. As before, the jump parameter η is determined to match the mean and variance of y_{t+1}.

The mean and variance of y_{t+1} given $\left(y_t, h_t^2\right)$ at day t are

$$\begin{aligned} \mathrm{E}_t\left[y_{t+1}\right] &= y_t + r - \frac{1}{2}h_t^2, \\ \mathrm{Var}_t\left[y_{t+1}\right] &= h_t^2, \end{aligned} \tag{18}$$

respectively. The sum of the probabilities for the up, middle, and down branches must equal 1:

$$p_u + p_m + p_d = 1. \tag{19}$$

The following probabilities for the up, middle and down branches result from equations (18) and (19):

$$p_u = \frac{nh_t^2 + \left(a\gamma_n - \mu\right)^2}{2n^2\eta^2\gamma_n^2} - \frac{a\gamma_n - \mu}{2n\eta\gamma_n},$$

$$p_m = 1 - \frac{nh_t^2 + \left(a\gamma_n - \mu\right)^2}{n^2\eta^2\gamma_n^2},$$

$$p_d = \frac{nh_t^2 + \left(a\gamma_n - \mu\right)^2}{2n^2\eta^2\gamma_n^2} + \frac{a\gamma_n - \mu}{2n\eta\gamma_n}.$$

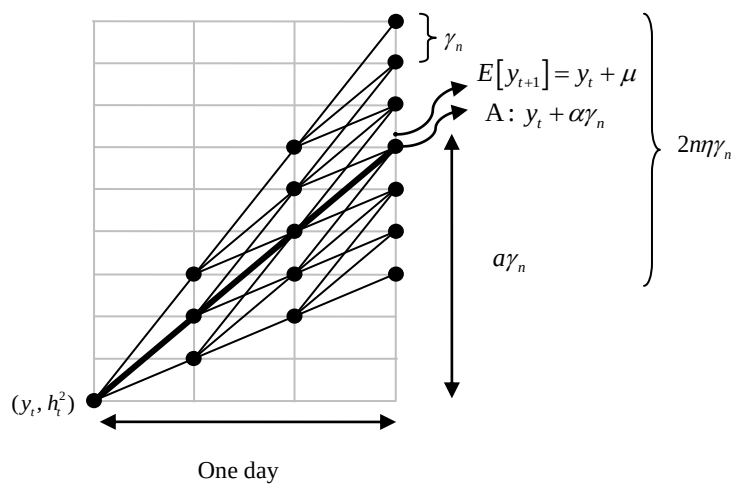


Fig. 3. The MT tree for one day. With $n = 3$, the MT tree becomes a 7-nomial tree after one day and spans over $2n\eta + 1$ nodes.

State $\left(y_t, h_t^2\right)$ at date t is followed by $\left(y_{t+1}, h_{t+1}^2\right)$ at date $t+1$, where

$$y_{t+1} = y_t + j\eta\gamma_n,$$

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 \left(\xi_{t+1} - c - \lambda\right)^2,$$

$$\xi_{t+1} = \frac{j\eta\gamma_n + a\gamma_n - \left(r - \frac{h_t^2}{2}\right)}{h_t},$$

$$j = 0, \pm 1, \pm 2, ..., \pm n.$$

The probabilities for the up, middle and down branches must satisfy

$$0 \le p_u,\ p_m,\ p_d \le 1. \tag{20}$$

Inequalities (16) and (20) together imply

$$\frac{\sqrt{nh_t^2 + \left(a\gamma_n - \mu\right)^2}}{n\gamma_n} \le \eta \le \frac{nh_t^2 + \left(a\gamma_n - \mu\right)^2}{n\gamma_n \left|a\gamma_n - \mu\right|}. \tag{21}$$

The above inequalities contain at least one positive integer if

$$\gamma_n^2 \le H_{\min}^2, \tag{22}$$

where

$$H_{\min}^2 = \min\left(h_0^2, \frac{\beta_0}{1-\beta_1}\right) \le h_t^2 \text{ for } t \ge 0. \tag{23}$$

Therefore, instead of searching for the jump parameter η as in RT, the jump parameter in MT is simply chosen as

$$\eta = \left\lceil \frac{\sqrt{nh_t^2 + \left(a\gamma_n - \mu\right)^2}}{n\gamma_n} \right\rceil. \tag{24}$$

As γ_n has to satisfy inequality (22), MT sets it to

$$\gamma_n = \frac{H_{\min}}{2\sqrt{n}}. \tag{25}$$

Inequality (16) and formula (25) together imply

$$\eta \le \frac{2h_t}{H_{\min}} + 2, \tag{26}$$

which will be used in deriving the non-explosion thresholds for the other 5 GARCH models later. For more detailed analysis, see Lyuu and Wu [7].

5. NGARCH

As the explosion and non-explosion thresholds for NGARCH are detailed in Section 1 and Lyuu and Wu [7], we only review the results here.

5.1 The Explosion Threshold

In RT, h_t^2 under NGARCH follows

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 \left(\xi_{t+1} - c - \lambda\right)^2,$$

where

$$\xi_{t+1} = \frac{j\eta\gamma_n - \left(r - \frac{h_t^2}{2}\right)}{h_t},$$

$$j = 0, \pm 1, \pm 2, ..., \pm n.$$

The largest value of h_t^2 at date t grows exponentially if

$$\beta_1 + \beta_2 n > 1. \tag{27}$$

Inequality (27) is the explosion threshold. When h_t^2 grows exponentially, the RT tree explodes by virtue of inequality (13).

5.2 The Non-explosion Threshold

With MT, h_t^2 under NGARCH follows

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 \left(\xi_{t+1} - c - \lambda\right)^2,$$

where

$$\xi_{t+1} = \frac{j\eta\gamma_n + a\gamma_n - \left(r - \frac{h_t^2}{2}\right)}{h_t},$$

$$j = 0, \pm 1, \pm 2, ..., \pm n.$$

The largest value of h_t^2 at date t does not grow exponentially if

$$\beta_1 + \beta_2 \left(\sqrt{n} + c + \lambda\right)^2 \le 1. \tag{28}$$

Inequality (28) is the non-explosion threshold. As long as the above inequality holds, the MT tree does not explode; in fact, its number of nodes grows only quadratically in *n*.

5.3 Numerical Results under the Explosion Condition

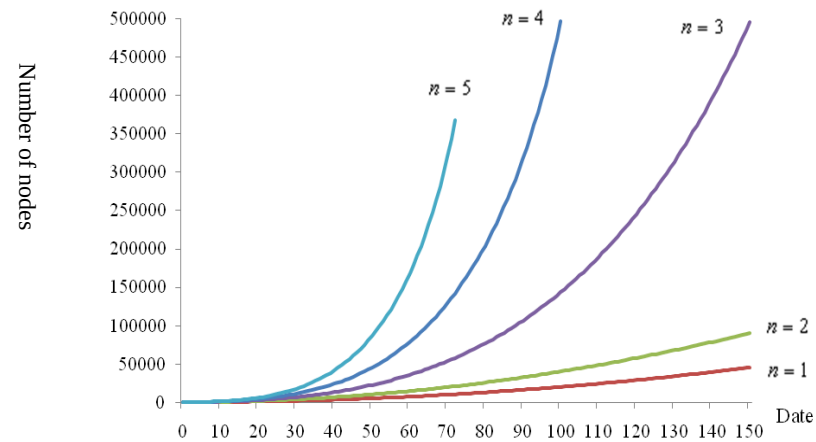
The parameters are as follows: $S_0 = 100$, $r = 0$, $h_0^2 = 0.0001096$, $\beta_0 = 0.000006575$, $\beta_1 = 0.9$, $\beta_2 = 0.04$, $\beta_3 = 0.04$, $c = 0$, $K = 3$ and $\lambda = 0$. The maturity is fixed at 150 days throughout this paper. This group of parameters will be used from Sections 5 to 10 in the explosion conditions, including NGARCH, LGARCH, AGARCH, GJR-GARCH, TS-GARCH, TGARCH unless stated otherwise. These 6 GARCH models have similar characteristics in that both explosion and non-explosion thresholds depend on *n*.

The explosion threshold for NGARCH is $\beta_1 + \beta_2 n > 1$ by inequality (27), so explosion occurs when

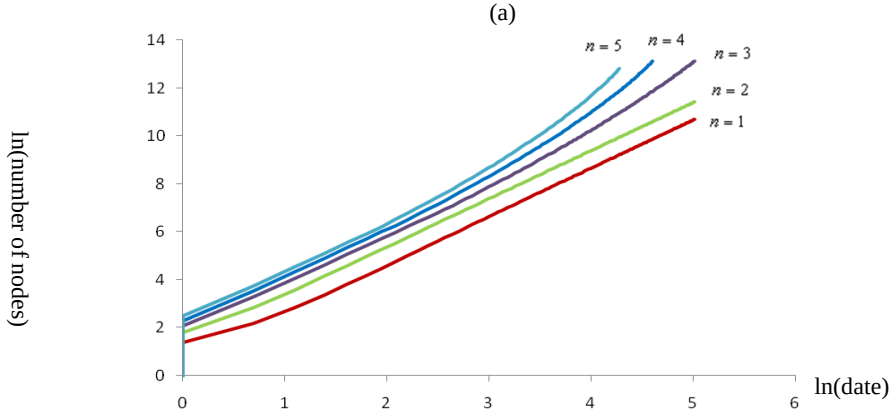
$$n > \frac{1 - \beta_1}{\beta_2} = 2.5.$$

Fig. 4 presents the experimental results for $n = 1, 2, 3, 4, 5$. With $n = 3, 4, 5$, the RT

trees explode. Furthermore, the RT trees with $n = 4, 5$ are cut short, respectively, at date 102 and date 74. With $n = 3$, the RT tree is not cut short before the maturity, but the theory predicts it will be at a later maturity. With $n = 1, 2$, the theory is silent about whether the RT trees explode although they do not seem to.



(a)



(b)

Fig. 4. The explosion threshold for NGARCH. The curves with $n = 3, 4, 5$ satisfy the explosion threshold, whereas the curves with $n = 1, 2$ do not. The log-log plot appears in (b).

5.4 Numerical Results under the Non-explosion Condition

The parameters are as follows: $S_0 = 100$, $r = 0$, $h_0^2 = 0.0001096$, $\beta_0 = 0.000006575$, $\beta_1 = 0.9$, $\beta_2 = 0.04$, $\beta_3 = 0.04$, $c = 0.04$, $K = 3$ and

$\lambda = 0.04$. This group of parameters will be used from Sections 5 to 10 for the non-explosion conditions under unless stated otherwise.

The threshold for non-explosion for NGARCH is $\beta_1 + \beta_2\left(\sqrt{n} + c + \lambda\right)^2 \le 1$ by inequality (28), so MT tree does not explode when

$$n \le \left(\sqrt{\frac{1-\beta_1}{\beta_2}} - c - \lambda\right)^2 = \left(\sqrt{\frac{1-0.9}{0.04}} - 0.04 - 0.04\right)^2 = 2.253.$$

Fig. 5 presents the experimental results for $n = 2, 3, 4$. With $n = 2$, the MT tree does not explode. But with $n = 3, 4$, the theory is silent about whether the MT trees explode.

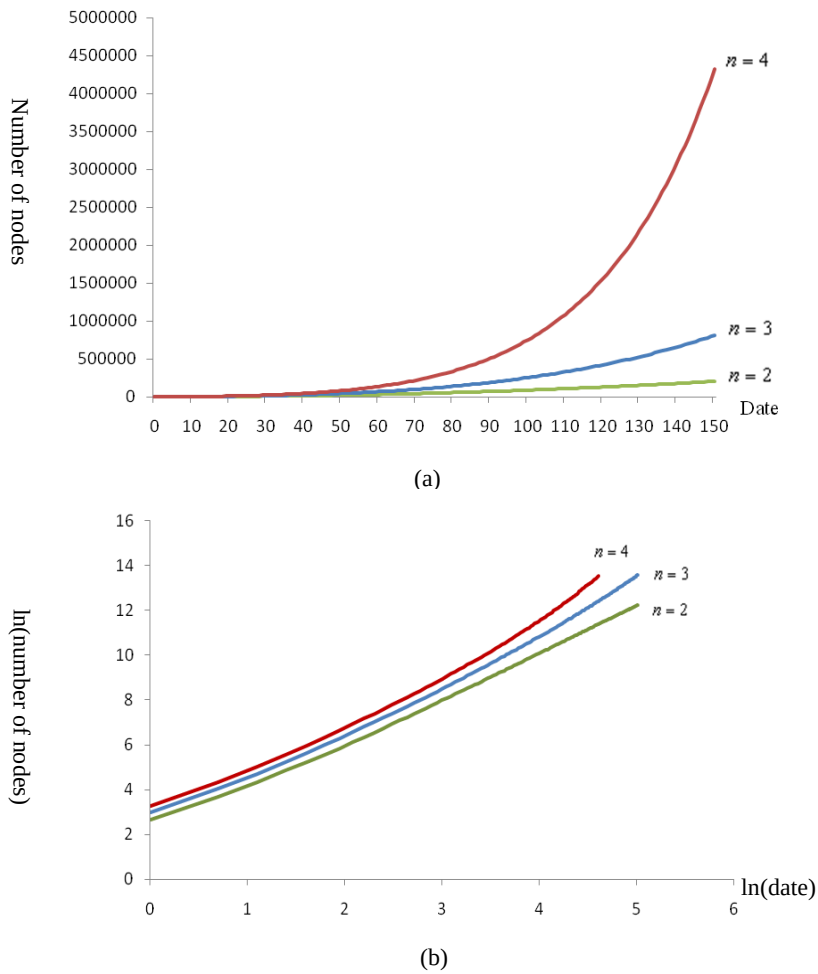


Fig. 5. The non-explosion threshold for NGARCH. The curve with $n = 2$ satisfies the non-explosion threshold, whereas the curves with $n = 3, 4$ do not. The log-log plot appears in (b).

6. LGARCH

6.1 The Explosion Threshold

LGARCH is identical to NGARCH except for the extra parameter c in NGARCH. As NGARCH contains LGARCH as a special case, the same explosion threshold holds for LGARCH. Therefore, the largest value of h_t^2 at date t grows exponentially if

$$\beta_1 + \beta_2 n > 1. \tag{29}$$

Inequality (29) is the explosion threshold. When h_t^2 grows exponentially, the RT tree explodes by inequality (13).

6.2 The Non-explosion Threshold

The only difference between LGARCH and NGARCH is the extra c in NGARCH. It is straightforward to obtain the same non-explosion threshold (30) by following the same proof. Therefore, the largest value of h_t^2 at date t does not grow exponentially if

$$\beta_1 + \beta_2 \left(\sqrt{n} + \lambda\right)^2 \le 1. \tag{30}$$

Inequality (30) is the non-explosion threshold. As long as the above inequality holds, the MT tree does not explode. In fact, the number of nodes in the MT tree grows only quadratically in n.

6.3 Numerical Results under the Explosion Condition

LGARCH and NGARCH are identical except for the extra parameter c in NGARCH. With the same group of parameters, we get the same result, i.e., explosion occurs when

$$n > \frac{1-\beta_1}{\beta_2} = 2.5$$

(recall Fig. 4). The RT trees explode with $n = 3, 4, 5$. Furthermore, the RT trees with $n = 4$ and $n = 5$ are cut short, respectively, at date 102 and date 74. With $n = 3$, the RT tree is not cut short before the maturity, but the theory predicts it will be at a later maturity. With $n = 1, 2$, the theory is silent about whether the RT trees explode although they do not seem to.

6.4 Numerical Results under the Non-explosion Condition

The non-explosion threshold for NGARCH is $\beta_1 + \beta_2 \left(\sqrt{n} + \lambda\right)^2 \le 1$ by inequality (30), so the MT tree does not explode if

$$n \le \left(\sqrt{\frac{1-\beta_1}{\beta_2}} - \lambda\right)^2 = 2.5.$$

Fig. 6 presents the experimental results for $n = 2, 3, 4$. With $n = 2$, the MT tree does not explode. But with $n = 3, 4$, the theory is silent about whether the MT trees explode.

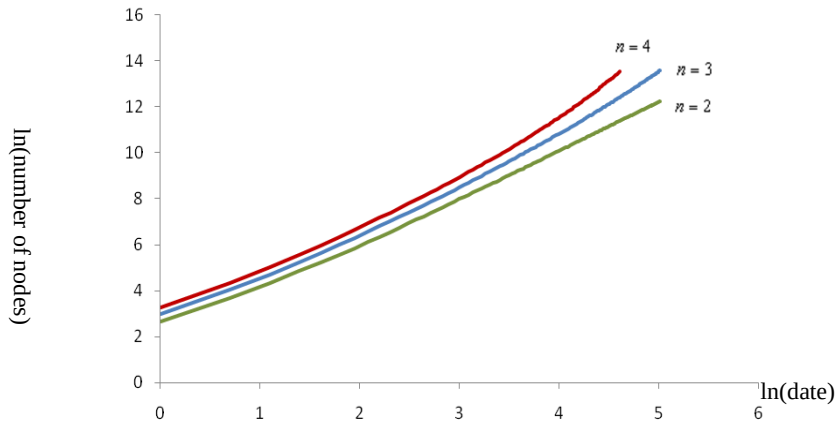


Fig. 6. The non-explosion threshold for LGARCH. The log-log plot shows that the curve with $n=2$ satisfies the non-explosion threshold, whereas the curves with $n=3,4$ do not.

7. AGARCH

7.1 The Explosion Threshold

In RT, h_t^2 under AGARCH follows

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta_2 \left(\xi_{t+1} h_t - \lambda h_t + c\right)^2,$$

where

$$\xi_{t+1} = \frac{j\eta\gamma_n - \left(r - \frac{h_t^2}{2}\right)}{h_t},$$

$$j = 0, \pm 1, \pm 2, \ldots, \pm n.$$

The largest value of h_t^2 at date t grows exponentially if

$$\beta_1 + \beta_2 n > 1. \tag{31}$$

Inequality (31) is the explosion threshold. When h_t^2 grows exponentially, the RT tree explodes by inequality (13). We proceed to prove the above claims.

Set $r=0$, $\lambda=0$ for simplicity for the rest of the subsection. The explosion threshold remains true for general r and λ if n is large enough. With $r=0$, $\lambda=0$ and $c=0$, AGARCH reduces to LGARCH; as a result, the same explosion threshold as that for LGARCH obtains for AGARCH.

7.2 The Non-explosion Threshold

In MT, h_t^2 under AGARCH follows

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta_2 \left(\xi_{t+1} h_t - \lambda h_t + c\right)^2,$$

where

$$\xi_{t+1} = \frac{j\eta\gamma_n + a\gamma_n - \left(r - \frac{h_t^2}{2}\right)}{h_t},$$

$$j = 0, \pm 1, \pm 2, ..., \pm n.$$

The largest value of h_t^2 at date t does not grow exponentially if

$$\beta_1 + \beta_2 \left(\sqrt{n} + \lambda\right)^2 \le 1. \tag{32}$$

Inequality (32) is the non-explosion threshold. As long as the above inequality holds, the MT tree does not explode. In fact, the number of nodes in the MT tree grows only quadratically in *n*. We proceed to prove the above claims.

Start with h_{t+1}^2 at date $t+1$:

$$\begin{aligned}
h_{t+1}^2 &= \beta_0 + \beta_1 h_t^2 + \beta_2 \left(\xi_{t+1} h_t - \lambda h_t + c\right)^2 \\
&= \beta_0 + \beta_1 h_t^2 + \beta_2 \left(j\eta\gamma_n + a\gamma_n - \left(r - \frac{h_t^2}{2}\right) - \lambda h_t + c \right)^2 \\
&\le \beta_0 + \beta_1 h_t^2 + \beta_2 \left(\left| j\eta\gamma_n \right| + \left| a\gamma_n - \mu \right| + \left| \lambda h_t \right| + \left| c \right| \right) \quad \text{by equation (15),}
\end{aligned}$$

where $j = 0, \pm 1, \pm 2, ..., \pm n.$ Define $A \equiv \sqrt{\beta_1 + \beta_2 \left(\sqrt{n} + \lambda\right)^2}$. Let $j = n$ to make the upper bound on h_{t+1}^2 as large as possible. Then

$$\begin{aligned}
h_{t+1}^2 &\le \beta_0 + \beta_1 h_t^2 + \beta_2 \left(n\eta\gamma_n + \frac{\gamma_n}{2} + \lambda h_t + c \right)^2 \quad \text{by inequality (16)} \\
&\le \beta_0 + \beta_1 h_t^2 + \beta_2 \left(\left(\frac{2nh_t}{H_{\min}} + 2n + \frac{1}{2} \right) \gamma_n + \lambda h_t + c \right)^2 \quad \text{by inequality (26)} \\
&\le \beta_0 + \beta_1 h_t^2 + \beta_2 \left(\left(\frac{2nh_t}{H_{\min}} + 2n + \frac{1}{2} \right) \frac{H_{\min}}{2\sqrt{n}} + \lambda h_t + c \right)^2 \quad \text{by equation (25)} \\
&= \beta_0 + \beta_1 h_t^2 + \beta_2 \left(\sqrt{n} h_t + \left(2n + \frac{1}{2} \right) \frac{H_{\min}}{2\sqrt{n}} + \lambda h_t + c \right)^2 \\
&\le \beta_0 + \beta_1 h_t^2 + \beta_2 \left(\left(\sqrt{n} + \lambda\right) h_t + C \right)^2 \\
&= \left(\beta_1 + \beta_2 \left(\sqrt{n} + \lambda\right)^2 \right) h_t^2 + 2\beta_2 C \left(\sqrt{n} + \lambda\right) h_t + \beta_0 + \beta_2 C^2 \\
&= \left(A h_t + B\right)^2 + E,
\end{aligned}$$

where B, C and E are positive numbers independent of h_t and t. Hence there exists a number $D>0$ independent of h_t and $h_0^2 = 0.0001096$ such that

$$h_{t+1}^2 \le \left(A h_t + B\right)^2 + E \le \left(A h_t + D\right)^2,$$

which yields

$$h_{t+1} \le Ah_t + D.$$

Let H_t stand for the largest of all the volatilities at date t. Assume H_{t+1} at date $t+1$ follows volatility $\tilde{h}_t$ at date t. We obtain

$$H_{t+1} \le A\tilde{h}_t + D \le AH_t + D.$$

By induction,

$$H_{t+1} \le D\sum_{i=0}^{t} A^i + A^{t+1}h_0 = \frac{D}{1-A} + \left(h_0 + \frac{D}{A-1}\right) A^{t+1}.$$

Therefore, H_{t+1} at date $t+1$ does not grow exponentially when $A \le 1$, i.e.,

$$\beta_1 + \beta_2\left(\sqrt{n}+\lambda\right)^2 \le 1.$$

Given that the above inequality holds,

$$H_{t+1} \le \frac{D}{1-A} + \left(h_0 + \frac{D}{A-1}\right) A^{t+1} \le \frac{D}{1-A} + h_0$$

because $A-1 \le 0$.

As the top and bottom nodes span over $2n\eta+1$ nodes and inequality (26) holds, the number of nodes at date t is

$$\sum_{i=0}^{t-1}\left(2n\left(\frac{2H_t}{H_{\min}}+2\right)+1\right) = (4n+1)t + \frac{4n}{H_{\min}}\sum_{i=0}^{t-1} H_t$$

$$\le (4n+1)t + \frac{4n}{H_{\min}} t\left(\frac{D}{1-A} + h_0\right).$$

Therefore the total number of nodes of an *N*-day MT tree is at most

$$1 + \sum_{t=1}^{N}\left(4n+1+\frac{4n}{H_{\min}}\left(\frac{D}{1-A}+h_0\right)\right)t,$$

a quadratic growth, so the MT tree does not explode.

7.3 Numerical Results under the Explosion Condition

AGARCH and LGARCH are identical when $c=0$, so explosion occurs when

$$n > \frac{1-\beta_1}{\beta_2} = 2.5,$$

(recall Fig. 4). The RT trees with $n=4$ and $n=5$ explode and are cut short, respectively, at date 102 and date 74. With $n=3$, the RT tree also explodes, but is not cut short before the maturity although it will at a later maturity. With $n=1,2$, the theory is silent about whether if the RT trees explode although they do not seem to.

7.4 Numerical Results under the Non-explosion Condition

The non-explosion threshold for AGARCH is $\beta_1 + \beta_2\left(\sqrt{n}+\lambda\right)^2 \le 1$ by inequality (32), so the MT tree does not explode if

$$n \le \left(\sqrt{\frac{1-\beta_1}{\beta_2}} - \lambda\right)^2 = \left(\sqrt{\frac{1-0.9}{0.04}} - 0.04\right)^2 = 2.375.$$

Fig. 7 presents the experimental results for $n = 2, 3, 4$. With $n = 2$, the MT tree does not explode. But with $n = 3, 4$, the theory is silent about whether the MT trees explode.

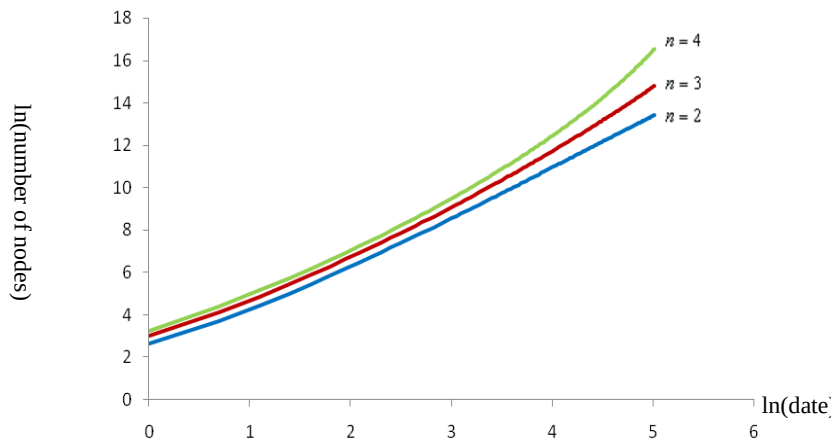


Fig. 7. The non-explosion threshold for AGARCH. The log-log plot shows that the curve with $n = 2$ satisfies the non-explosion threshold, whereas the curves with $n = 3, 4$ do not.

8. GJR-GARCH

8.1 The Explosion Threshold

In RT, h_t^2 under GJR-GARCH follows

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta h_t^2 \left(\xi_{t+1} - \lambda\right)^2 + \beta_3 h_t^2 \max\left(0, -\left(\xi_{t+1} - \lambda\right)\right)^2,$$

where

$$\xi_{t+1} = \frac{j\eta\gamma_n - \left(r - \frac{h_t^2}{2}\right)}{h_t},$$

$$j = 0, \pm 1, \pm 2, \ldots, \pm n.$$

The largest value of h_t^2 at date t grows exponentially if

$$\beta_1 + \beta_2 n > 1. \tag{33}$$

Inequality (33) is the explosion threshold. When h_t^2 grows exponentially, the RT tree explodes by inequality (13). We proceed to prove the above claims.

Start with h_{t+1}^2 at date $t+1$:

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 \left(\xi_{t+1} - \lambda\right)^2 + \beta_3 h_t^2 \max\left(0, -\left(\xi_{t+1} - \lambda\right)\right)^2$$

$$\geq \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 \left(\xi_{t+1} - \lambda\right)^2,$$

where $j = 0, \pm 1, \pm 2, \ldots, \pm n.$ This form is similar to LGARCH. By setting $\lambda = 0$ and

$r = 0$ and following the same proof as the one for LGARCH, we can conclude the largest value of h_t^2 at date t grows exponentially if

$$\beta_1 + \beta_2 n > 1.$$

8.2 The Non-explosion Threshold

In MT, h_t^2 under GJR-GARCH follows

$$h_{t+1}^2 = \beta_0 + \beta_1 h_t^2 + \beta h_t^2 \left(\xi_{t+1} - \lambda\right)^2 + \beta_3 h_t^2 \max\left(0, -\left(\xi_{t+1} - \lambda\right)\right)^2,$$

where

$$\xi_{t+1} = \frac{j\eta\gamma_n + a\gamma_n - \left(r - \frac{h_t^2}{2}\right)}{h_t},$$

$$j = 0, \pm 1, \pm 2, ..., \pm n.$$

The largest value of h_t^2 at date t does not grow exponentially if

$$\beta_1 + \left(\beta_2 + \beta_3\right)\left(\sqrt{n} + \lambda\right)^2 \le 1. \tag{34}$$

Inequality (34) is the non-explosion threshold. As long as the above inequality holds, the MT tree does not explode. In fact, the number of nodes in the MT tree grows only quadratically in *n*. We proceed to prove the above claims.

The variance at date $t+1$ is

$$\begin{aligned} h_{t+1}^2 &= \beta_0 + \beta_1 h_t^2 + \beta h_t^2 \left(\xi_{t+1} - \lambda\right)^2 + \beta_3 h_t^2 \max\left(0, -\left(\xi_{t+1} - \lambda\right)\right)^2 \\ &\le \beta_0 + \beta_1 h_t^2 + \beta_2 h_t^2 \left(\xi_{t+1} - \lambda\right)^2 + \beta_3 h_t^2 \left(\xi_{t+1} - \lambda\right)^2 \\ &= \beta_0 + \beta_1 h_t^2 + \left(\beta_2 + \beta_3\right) h_t^2 \left(\xi_{t+1} - \lambda\right)^2, \end{aligned}$$

where $j = 0, \pm 1, \pm 2, ..., \pm n.$ This form is identical to LGARCH, so we just substitute β_2 in LGARCH with $\beta_2 + \beta_3$ to obtain the non-explosion threshold:

$$\beta_1 + \left(\beta_2 + \beta_3\right)\left(\sqrt{n} + \lambda\right)^2 \le 1.$$

8.3 Numerical Results under the Explosion Condition

As GJR-GARCH and LGARCH share the same explosion threshold (compare inequalities (27) and (33)), it is not surprising that they share similar numerical results as well. Explosion occurs when

$$n > \frac{1 - \beta_1}{\beta_2} = 2.5,$$

Fig. 8 presents the experimental results for $n = 1, 2, 3, 4, 5$. The RT trees explode with $n = 3, 4, 5$ and are cut short, respectively, at date 86, 59, and 46. But with $n = 1, 2$, the theory is silent about whether the RT trees explode although it seems to explode with $n = 2$ and does not with $n = 1$.

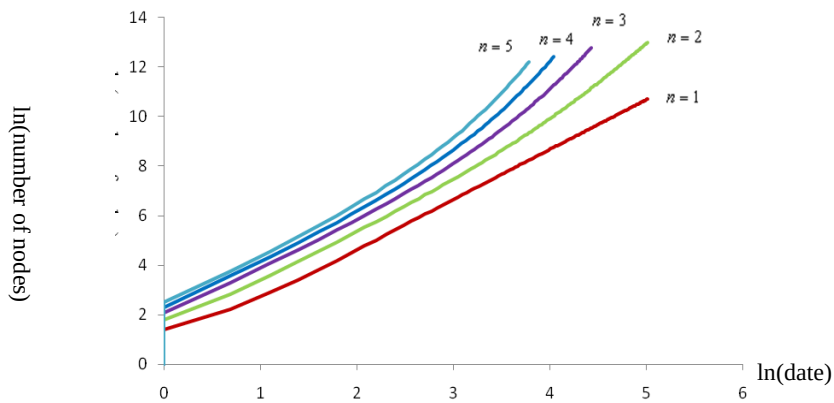


Fig. 8. The explosion threshold for GJR-GARCH. The log-log plot shows that the curves with $n = 3, 4, 5$ satisfy the explosion threshold, whereas the curves with $n = 1, 2$ do not.

8.4 Numerical Results under the Non-explosion Condition

The non-explosion threshold for GJR-GARCH is $\beta_1 + (\beta_2 + \beta_3)\left(\sqrt{n} + c\right)^2 \le 1$ by inequality (34), so the MT tree does not explode when

$$n > \left(\sqrt{\frac{1-\beta_1}{\beta_2+\beta_3}} - \lambda\right)^2 = \left(\sqrt{\frac{1-0.9}{0.04+0.04}} - 0.04\right)^2 = 1.162.$$

Fig. 9 presents the experimental results for $n = 1, 2$. With $n = 1$, the MT tree does not explode. But with $n = 2$, the theory is silent about whether the MT tree explodes although it seems to.

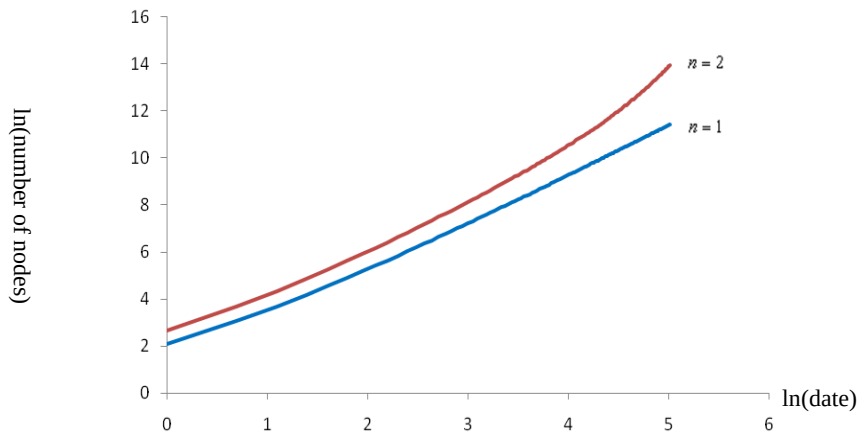


Fig. 9. The non-explosion threshold for GJR-GARCH. The log-log plot shows that the curve with $n=1$ satisfies the non-explosion threshold, whereas the curve with $n=2$ does not.

9. TS-GARCH

9.1 The Explosion Threshold

In RT, h_t under TS-GARCH follows

$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t \left|\xi_{t+1} - \lambda\right|,$$

where

$$\xi_{t+1} = \frac{j\eta\gamma_n - \left(r - \frac{h_t^2}{2}\right)}{h_t},$$

$$j = 0, \pm 1, \pm 2, \ldots, \pm n.$$

The largest value of h_t at date t grows exponentially if

$$\beta_1 + \beta_2\sqrt{n} > 1. \tag{35}$$

Inequality (35) is the explosion threshold. When h_t grows exponentially, the RT tree explodes by inequality (13). We proceed to prove the above claims.

With $r=0$ and $\lambda=0$, the volatility at date $t+1$ is

$$\begin{aligned} h_{t+1} &= \beta_0 + \beta_1 h_t + \beta_2 h_t \left|\xi_{t+1} - \lambda\right| \\ &= \beta_0 + \beta_1 h_t + \beta_2 h_t \left|\frac{j\eta\gamma_n + \frac{h_t^2}{2}}{h_t}\right| \end{aligned}$$

where $j = 0, \pm 1, \pm 2, ..., \pm n.$

Let $j = n$ to make the upper bound on h_{t+1} as large as possible. Now,

$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t \left| \frac{\sqrt{n}\eta\gamma + \frac{h_t^2}{2}}{h_t} \right|$$

$$\geq \beta_0 + \beta_1 h_t + \beta_2 h_t \left(\sqrt{n} + \frac{h_t}{2} \right) \qquad \text{by inequality (13)}$$

$$\geq \beta_0 + \left(\beta_1 + \beta_2 \sqrt{n} \right) h_t .$$

By induction,

$$h_{t+1} \geq \beta_0 \sum_{i=0}^{t} \left(\beta_1 + \beta_2 \sqrt{n} \right)^i + \left(\beta_1 + \beta_2 \sqrt{n} \right)^{t+1} h_0^2$$

$$= \frac{\beta_0}{1 - \left(\beta_1 + \beta_2 \sqrt{n} \right)} + \left[h_0^2 + \frac{\beta_0}{\left(\beta_1 + \beta_2 \sqrt{n} \right) - 1} \right] \left(\beta_1 + \beta_2 \sqrt{n} \right)^{t+1} .$$

The above expression grows exponentially in t if

$$\beta_1 + \beta_2 \sqrt{n} > 1.$$

9.2 The Non-explosion Threshold

In MT, h_t under TS-GARCH follows

$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t \left| \xi_{t+1} - \lambda \right|,$$

where

$$\xi_{t+1} = \frac{j\eta\gamma_n + a\gamma_n - \left(r - \frac{h_t^2}{2} \right)}{h_t},$$

$$j = 0, \pm 1, \pm 2, ..., \pm n.$$

The largest value of h_t at date t does not grow exponentially if

$$\beta_1 + \beta_2 \left(\lambda + \sqrt{n} \right) \leq 1. \tag{36}$$

Inequality (36) is the non-explosion threshold. As long as the above inequality holds, the MT tree does not explode. In fact, the number of nodes in the MT tree grows only quadratically in *n*. We proceed to prove the above claims.

The volatility at date $t+1$ is

$$
\begin{aligned}
h_{t+1} &= \beta_0 + \beta_1 h_t + \beta_2 h_t \left|\xi_{t+1} - \lambda\right| \\
&= \beta_0 + \beta_1 h_t + \beta_2 h_t \left| \frac{j\eta\gamma_n + a\gamma_n - \left(r - \frac{h_t^2}{2}\right)}{h_t} - \lambda \right| \\
&= \beta_0 + \beta_1 h_t + \beta_2 h_t \left| \frac{j\eta\gamma_n + a\gamma_n - \mu}{h_t} - \lambda \right| \quad \text{by equation (15),}
\end{aligned}
$$

where $j = 0, \pm 1, \pm 2, \ldots, \pm n.$

Let $j = n$ to make the upper bound on h_{t+1} as large as possible. Then

$$
\begin{aligned}
h_{t+1} &\le \beta_0 + \beta_1 h_t + \beta_2 \left(n\eta\gamma_n + \left|a\gamma_n - \mu\right| + \lambda h_t \right) \\
&\le \beta_0 + \beta_1 h_t + \beta_2 \left(n\eta\gamma_n + \frac{\gamma_n}{2} + \lambda h_t \right) \quad \text{by inequality (16)} \\
&\le \beta_0 + \left(\beta_1 + \beta_2\lambda\right) h_t + \beta_2 \left(\frac{2nh_t}{H_{\min}} + 2n + \frac{1}{2} \right) \gamma_n \quad \text{by inequality (26)} \\
&= \beta_0 + \left(\beta_1 + \beta_2\lambda\right) h_t + \beta_2 \left(\sqrt{n} h_t + \left(2n + \frac{1}{2}\right) \frac{H_{\min}}{2\sqrt{n}} \right) \quad \text{by equation (25)} \\
&= \beta_0 + \beta_2 \left(2n + \frac{1}{2}\right) \frac{H_{\min}}{2\sqrt{n}} + \left(\beta_1 + \beta_2\left(\lambda + \sqrt{n}\right)\right) h_t .
\end{aligned}
$$

Let H_t stand for the largest of all the volatilities at date t. Now,

$$
\begin{aligned}
H_{t+1} &\le \left(\beta_0 + \beta_2 \left(2n + \frac{1}{2}\right) \frac{H_{\min}}{2\sqrt{n}} \right) \sum_{i=0}^{t} \left(\beta_1 + \beta_2\left(\lambda + \sqrt{n}\right)\right)^i + \left(\beta_1 + \beta_2\left(\lambda + \sqrt{n}\right)\right)^{t+1} h_0 \\
&= \frac{\beta_0 + \beta_2 \left(2n + \frac{1}{2}\right) \frac{H_{\min}}{2\sqrt{n}}}{1 - \left(\beta_1 + \beta_2\left(\lambda + \sqrt{n}\right)\right)} + \left(h_0 + \frac{\beta_0 + \beta_2 \left(2n + \frac{1}{2}\right) \frac{H_{\min}}{2\sqrt{n}}}{\left(\beta_1 + \beta_2\left(\lambda + \sqrt{n}\right)\right) - 1} \right) \left(\beta_1 + \beta_2\left(\lambda + \sqrt{n}\right)\right)^{t+1} .
\end{aligned}
$$

Therefore, H_{t+1} at date $t+1$ does not grow exponentially when

$$\beta_1 + \beta_2\left(\lambda + \sqrt{n}\right) \le 1.$$

Given that the above inequality holds,

$$
H_{t+1} \le \frac{\beta_0 + \beta_2 \left(2n + \frac{1}{2}\right) \frac{H_{\min}}{2\sqrt{n}}}{1 - \left(\beta_1 + \beta_2\left(\lambda + \sqrt{n}\right)\right)} + h_0 .
$$

As the top and bottom nodes span over $2n\eta + 1$ nodes and inequality (26) holds, the number of nodes at date t is

$$\sum_{i=0}^{t-1}\left(2n\left(\frac{2H_t}{H_{\min}}+2\right)+1\right)=(4n+1)t+\frac{4n}{H_{\min}}\sum_{i=0}^{t-1}H_t$$

$$\leq(4n+1)t+\frac{4n}{H_{\min}}\left(\frac{\beta_0+\beta_2\left(2n+\frac{1}{2}\right)\frac{H_{\min}}{2\sqrt{n}}}{1-\left(\beta_1+\beta_2\left(\lambda+\sqrt{n}\right)\right)}+h_0\right)t$$

$$=\left(4n+1+\frac{4n}{H_{\min}}\left(\frac{\beta_0+\beta_2\left(2n+\frac{1}{2}\right)\frac{H_{\min}}{2\sqrt{n}}}{1-\left(\beta_1+\beta_2\left(\lambda+\sqrt{n}\right)\right)}+h_0\right)\right)t.$$

Therefore the total number of nodes of an N-day MT tree is at most

$$1+\sum_{t=1}^{N}\left(4n+1+\frac{4n}{H_{\min}}\left(\frac{\beta_0+\beta_2\left(2n+\frac{1}{2}\right)\frac{H_{\min}}{2\sqrt{n}}}{1-\left(\beta_1+\beta_2\left(\lambda+\sqrt{n}\right)\right)}+h_0\right)\right)t,$$

a quadratic growth, so the MT tree does not explode.

9.3 Numerical Results under the Explosion Condition

The explosion threshold for TS-GARCH is $\beta_1+\beta_2\left(\lambda+\sqrt{n}\right)>1$ by inequality (35), so the RT tree explodes when

$$n\geq\left(\frac{1-\beta_1}{\beta_2}-\lambda\right)^2=\left(\frac{1-0.9}{0.04}-0\right)^2=6.25.$$

Fig. 10 presents the experimental results for $n=6,9$. With $n=9$, the RT tree explodes. But with $n=6$, the theory is silent about whether the RT tree explodes although it does not seem to.

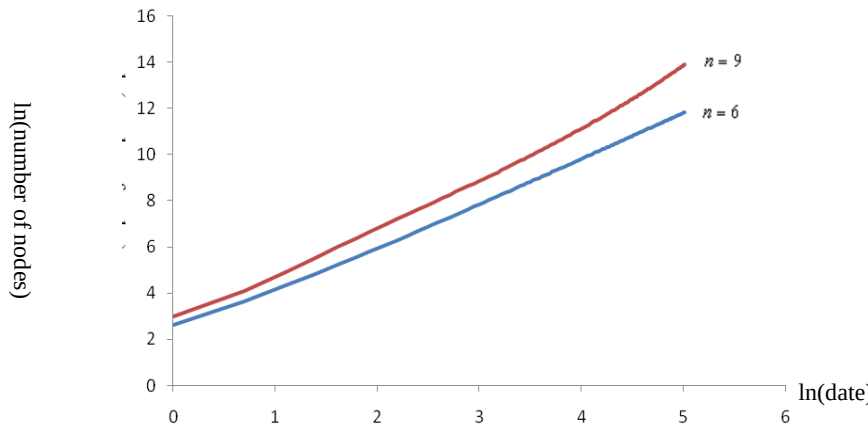


Fig. 10. The explosion threshold for TS-GARCH. The log-log plot shows that the curve with $n = 6$ satisfies the explosion threshold, whereas the curve with $n = 9$ does not.

9.4 Numerical Results under the Non-explosion Condition

The non-explosion threshold for TS-GARCH is $\beta_1 + \beta_2\left(\lambda + \sqrt{n}\right) \le 1$ by inequality (36), so the MT tree does not explode when

$$n \le \left(\frac{1-\beta_1}{\beta_2} - \lambda\right)^2 = \left(\frac{1-0.9}{0.04} - 0.04\right)^2 = 6.0516.$$

Fig. 11 presents the experimental results for $n = 3, 9$. With $n = 3$, the MT tree does not explode. But with $n = 9$, the theory is silent about whether the MT tree explodes although it seems to.

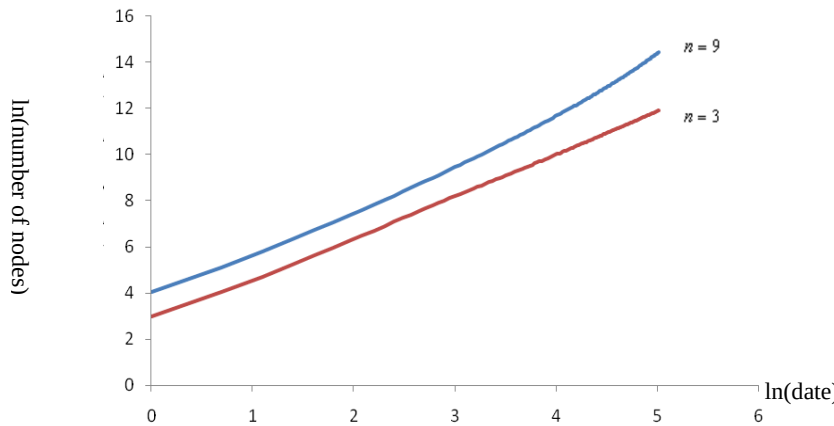


Fig. 11. The non-explosion threshold for TS-GARCH. The log-log plot shows that the curve with $n=3$ satisfies the non-explosion threshold, whereas the curve with $n=9$ does not.

10. TGARCH

10.1 The Explosion Threshold

In RT, h_t under TGARCH follows

$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t \left|\xi_{t+1} - \lambda\right| + \beta_3 h_t \max\left(0, -\left(\xi_{t+1} - \lambda\right)\right),$$

where

$$\xi_{t+1} = \frac{j\eta\gamma_n - \left(r - \frac{h_t^2}{2}\right)}{h_t},$$

$$j = 0, \pm 1, \pm 2, \ldots, \pm n.$$

The largest value of h_t at date t grows exponentially if

$$\beta_1 + \beta_2\sqrt{n} > 1. \tag{37}$$

Inequality (37) is the explosion threshold. When h_t grows exponentially, the RT tree explodes by inequality (13). We proceed to prove the above claims.

The volatility at date $t+1$ is

$$\begin{aligned} h_{t+1} &= \beta_0 + \beta_1 h_t + \beta_2 h_t \left|\xi_{t+1} - \lambda\right| + \beta_3 h_t \max\left(0, -\left(\xi_{t+1} - \lambda\right)\right) \\ &\geq \beta_0 + \beta_1 h_t + \beta_2 h_t \left|\xi_{t+1} - \lambda\right|, \end{aligned}$$

where $j = 0, \pm 1, \pm 2, \ldots, \pm n.$ This form is identical to TS-GARCH in Subsection 9.1. By setting $r=0$ and $\lambda = 0$, we obtain the same explosion threshold:

$$\beta_1 + \beta_2\sqrt{n} > 1.$$

10.2 The Non-explosion Threshold

In MT, h_t under TGARCH follows

$$h_{t+1} = \beta_0 + \beta_1 h_t + \beta_2 h_t \left|\xi_{t+1} - \lambda\right| + \beta_3 h_t \max\left(0, -\left(\xi_{t+1} - \lambda\right)\right),$$

where

$$\xi_{t+1} = \frac{j\eta\gamma_n + a\gamma_n - \left(r - \frac{h_t^2}{2}\right)}{h_t},$$

$$j = 0, \pm 1, \pm 2, ..., \pm n.$$

The largest value of h_t at date t does not grow exponentially if

$$\beta_1 + \left(\beta_2 + \beta_3\right)\left(\lambda + \sqrt{n}\right) \le 1. \tag{38}$$

Inequality (38) is the non-explosion threshold. As long as the above inequality holds, the MT tree does not explode. In fact, the number of nodes in the MT tree grows only quadratically in *n*. We proceed to prove the above claims.

The volatility at date $t+1$ is

$$\begin{aligned} h_{t+1} &= \beta_0 + \beta_1 h_t + \beta_2 h_t \left|\xi_{t+1} - \lambda\right| + \beta_3 h_t \max\left(0, -\left(\xi_{t+1} - \lambda\right)\right) \\ &\le \beta_0 + \beta_1 h_t + \beta_2 h_t \left|\xi_{t+1} - \lambda\right| + \beta_3 h_t \left|\xi_{t+1} - \lambda\right| \\ &= \beta_0 + \beta_1 h_t + \left(\beta_2 + \beta_3\right) h_t \left|\xi_{t+1} - \lambda\right|, \end{aligned}$$

where $j = 0, \pm 1, \pm 2, ..., \pm n.$ This form is identical to TS-GARCH once β_2 in TS-GARCH is replaced by $\beta_2 + \beta_3$. Following the same proof as the one for TS-GARCH, we get the non-explosion threshold:

$$\beta_1 + \left(\beta_2 + \beta_3\right)\left(\lambda + \sqrt{n}\right) \le 1.$$

10.3 Numerical Results under the Explosion Condition

The explosion threshold for TGARCH is the same as the one for TS-GARCH, so the RT tree explodes when $n > 6.25$.

Fig. 12 presents the experimental results for $n = 5, 6, 7, 8$. The curves with $n = 7, 8$ explode and are cut short, respectively, at date 99 and date 81. But with $n = 5, 6$, the theory is silent about whether the RT trees explode although they seem to.

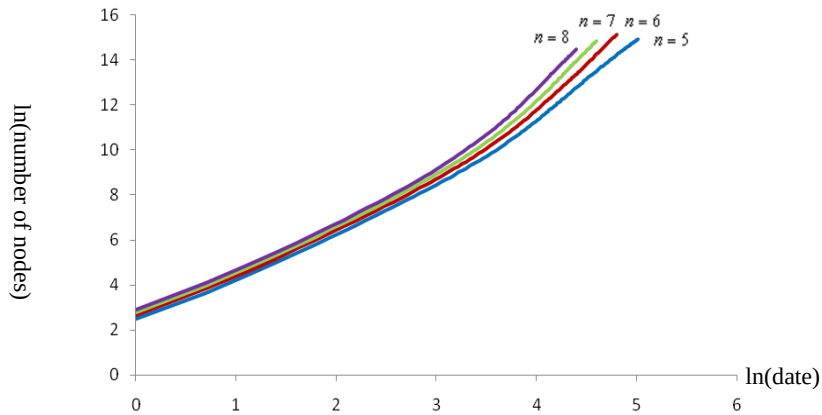


Fig. 12. The explosion threshold for TGARCH. The log-log plot shows that the curves with $n=7$ and $n=8$ satisfy the explosion threshold, whereas the curves with $n=5$ and $n=6$ do not.

10.4 Numerical Results under the Non-explosion Condition

The non-explosion threshold for TGARCH is $\beta_1+(\beta_2+\beta_3)\left(\lambda+\sqrt{n}\right)\le 1$ by inequality (38), so the MT tree does not explode when

$$n \le \left(\frac{1-\beta_1}{\beta_2+\beta_3}-\lambda\right)^2 = \left(\frac{1-0.9}{0.04+0.04}-0.04\right)^2 = 1.4641.$$

Fig. 13 presents the experimental results for $n=1,4$. With $n=1$, the MT tree does not explode. But with $n=4$, the theory is silent about whether the MT tree explodes although it seems to.

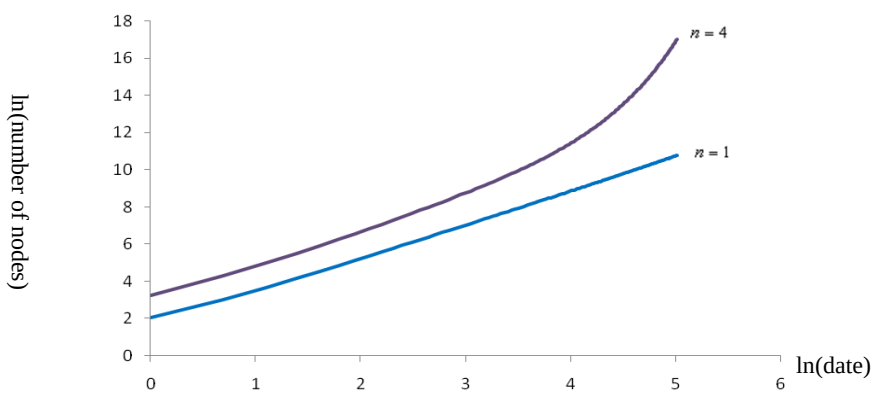


Fig. 13. The non-explosion threshold for TGARCH. The log-log plot shows that the curve with $n=1$ satisfies the non-explosion threshold, whereas the curve with $n=4$ does not.

11. Conclusion

Lyuu and Wu use NGARCH to derive the explosion threshold for RT and the non-explosion threshold for MT [7]. This paper derives explosion and non-explosion thresholds for 5 other GARCH models: LGARCH, AGARCH, GJR-GARCH, TS-GARCH, and TGARCH. The explosion and non-explosion thresholds for the 6 GARCH models are summarized in Table 1. When the RT tree explodes, the tree building must be cut off before a maturity date because of the issue of negative probabilities. When the MT tree does not explode, in contrast, its size is only quadratic. For NGARCH, LGARCH, AGARCH, GJR-GARCH, TS-GARCH and TGARCH, the explosion and non-explosion thresholds depend on *n*.

Table 1: The explosion and non-explosion threshold. Sufficient conditions for the 6 GARCH models to explode and not to explode are shown in the second column and the third column, respectively.

Model	Explosion threshold	Non-explosion threshold
NGARCH	$\beta_1 + \beta_2 n > 1$	$\beta_1 + \beta_2\left(\sqrt{n} + c + \lambda\right)^2 \le 1$
LGARCH	$\beta_1 + \beta_2 n > 1$	$\beta_1 + \beta_2\left(\sqrt{n} + \lambda\right)^2 \le 1$
AGARCH	$\beta_1 + \beta_2 n > 1$	$\beta_1 + \beta_2\left(\sqrt{n} + \lambda\right)^2 \le 1$
GJR-GARCH	$\beta_1 + \beta_2 n > 1$	$\beta_1 + \left(\beta_2 + \beta_3\right)\left(\sqrt{n} + \lambda\right)^2 \le 1$
TS-GARCH	$\beta_1 + \beta_2\sqrt{n} > 1$	$\beta_1 + \beta_2\left(\lambda + \sqrt{n}\right) \le 1$
TGARCH	$\beta_1 + \beta_2\sqrt{n} > 1$	$\beta_1 + \left(\beta_2 + \beta_3\right)\left(\lambda + \sqrt{n}\right) \le 1$

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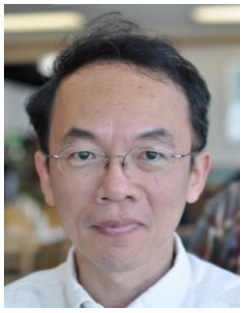
Ying-Chie Chen (陳盈潔) received her M.S. degree in finance from National Taiwan University, Taipei, Taiwan, in 2009. She is now working as a fixed-income analyst.



Yuh-Dauh Lyuu (呂育道) received his B.S. degree in Information Engineering from National Taiwan University, Taipei, Taiwan, in 1984 and his Ph.D. degree in Computer Science from Harvard University in 1990. He is now a Processor of Computer Science and Information Engineering Department, Finance Department, and the Graduate Institute of Networking and Multimedia, National Taiwan University, Taipei, Taiwan. His research interests include design and analysis of algorithms, theory of computation, and financial computation.

Kuo-Wei Wen (文國煒) received his B.S. and M.S. degrees in Computer Science and Information Engineering from National Cheng Kung University, Tainan, Taiwan, in 2003 and 2005, respectively. Currently, he is a Ph.D. candidate in the Department of Computer Science and Information Engineering, National Taiwan University, Taipei, Taiwan. His research interests include design and analysis of algorithms, and financial computation.