Travel Time Distances Induced by Transportation Networks and General Underlying Distances*

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Abstract

This paper considers a generalization of travel time distances by taking general underlying distance functions into account. We suggest a reasonable set of axioms defining a certain class of distance functions that can be facilitated with transportation networks. It turns out to be able to build an abstract framework for computing shortest path maps and Voronoi diagrams with respect to the induced travel time distance under such a general setting. We apply our framework in convex distance functions as a concrete example, resulting in efficient algorithms that compute the travel-time Voronoi diagram for a set of given sites. More specifically, the Voronoi diagram with respect to the travel-time distance induced by a convex distance based on a k-gon can be computed in $O(m(n+m)(k\log(n+m)+m))$ time and $O(km(n+m))$ space, where n is the number of Voronoi sites and m is the complexity of the given transportation network.

1 Introduction

Transportation networks model facilities for fast movement on the plane. In previous works, a transportation network is defined to be an edge-weighted graph whose edges are considered as road segments with specified speed under a certain underlying distance function, such as the Euclidean and the L_1 metric [1–4, 6–9, 13, 24]. In the presence of a transportation network with a certain rule of traveling, a shortest travel time path between any two points is well defined and the travel time distance measures the travel time along such a shortest path.

Since Abellabas et al. [1, 2] have first studied the shortest travel time path and the Voronoi diagram in the presence of a single highway, many researchers paid their attention to the subject. Aichholzer et al. [4] dealt with a transportation network with axis-parallel roads under the L_1 metric and considered the Voronoi diagram in the setting, namely the *city Voronoi diagram*. Subsequently, Görke and Wolff [13] came up with an improved algorithm and finally Bae et al. [8] presented an optimal algorithm that computes the city Voronoi diagram of a given set of sites. The authors also studied the travel time distance induced by a general transportation network under the Euclidean metric [6].

Related proximity problems have also been considered in the presence of a transportation network. Palop [24] studied fundamental proximity problems in her thesis other than the Voronoi diagram. Ahn et al. [3] considered several variants of problems of locating a highway

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or a road. Bae et al. [9] presented an efficient algorithm to the all-farthest-neighbor problem under travel time distances, and Bae and Chwa [7] studied the farthest-site Voronoi diagram.

These previous results, however, consider the Euclidean or the L_1 metric only as an underlying distance function. This paper discusses more general distance functions that can be facilitated with a transportation network, called *transportation admissible distance functions*, and travel time distances induced by such a general distance function. Our contributions are threefold:

- A class of transportation admissible distance functions is defined by a set of axioms. The class is shown to be quite wide, including all L_p metrics and symmetric or asymmetric convex distance functions.
- An algorithmic framework for computing shortest paths or Voronoi diagram with respect to the induced travel time distance is presented.
- Efficient algorithms that compute shortest paths and Voronoi diagrams are presented for travel time distance induced by a convex distance function, being either symmetric or asymmetric.

The organization of the paper is as follows: Section 2 defines the transportation admissible distance function, transportation networks, and related concepts. Section 3 is devoted to describe our algorithmic framework to computing shortest travel time paths and Voronoi diagrams with respect to the travel time distance, and we apply it to convex distance functions in Section 4. Section 5 finally concludes the paper.

2 Travel Time Distances under General Distances

In this section, we define transportation networks and travel time distances under general distance functions. Not all distance functions induce a proper travel time distance since it implicitly assumes paths realizing the distance itself. We first define a class of distance functions that admit transportation networks and thus travel time distances.

2.1 Distance Functions Admitting Transportation Networks

Let $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be a total distance function on $\mathbb{R}^2$. For the Euclidean and the L_1 metrics, a transportation network can be sufficiently represented as a planar straight-line graph. If, however, we consider more general distances, the meaning of "straight" should be reconsidered. Note that a straight segment is a shortest path or a geodesic on the Euclidean plane or on the L_1 plane. Geodesics, in general, naturally generalize straight segments, and a road can be defined to be a segment along a geodesic. Thus, in order to build a transportation network under d, d needs to admit a geodesic between any two points on the plane.

In this section, we discuss distance functions d that admit geodesics and transportation networks in an abstract and axiomatic fashion. We first impose d to be a *quasi-metric*.

Definition 1. A function $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is a *quasi-metric* if and only if the following axioms are fulfilled:

(i) $d(p, q) \geq 0$ for any $p, q \in \mathbb{R}^2$.
(ii) $d(p, q) = d(q, p) = 0$ if and only if $p = q$.
(iii) $d(p, q) \leq d(p, r) + d(r, q)$ for any $p, q, r \in \mathbb{R}^2$.

A quasi-metric is not necessarily symmetric; if a quasi-metric is symmetric, then it is a metric. A quasi-metric d induces two associated topologies by two families of open balls,

$B_d^+(x,\epsilon) = \{y \in \mathbb{R}^2 \mid d(x,y) < \epsilon\}$ and $B_d^-(x,\epsilon) = \{y \in \mathbb{R}^2 \mid d(y,x) < \epsilon\}$ for any $x \in \mathbb{R}^2$ and $\epsilon > 0$, called *forward* and *backward*, respectively [15, 23]. Here, we consider only the backward case as we shall take only the *inward* Voronoi diagrams into account where Voronoi sites are regarded as stationary sites or servers, so as destinations of clients. In this view, we define *transportation admissible distances* as follows.

Definition 2. A function $d\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is *transportation admissible* if and only if the following axioms are fulfilled:

T1 d is a quasi-metric on $\mathbb{R}^2$.
T2 The family of backward open d-balls $B_d^-(x,\epsilon)$ induced by d on $\mathbb{R}^2$ induces the Euclidean topology.
T3 The backward d-balls $B_d^-(x,\epsilon)$ are bounded with respect to the Euclidean metric.
T4 For any two points $p, q \in \mathbb{R}^2$, there exists a point $r \in \mathbb{R}^2 \setminus \{p,q\}$ such that $d(p,r) + d(r,q) = d(p,q)$.

Axioms T2–T4 in the above definition indeed mimic those of *nice metrics* in the sense of Klein and Wood [19]. By the axioms, the quasi-metric space $(\mathbb{R}^2, d)$ induced by a transportation admissible distance d on $\mathbb{R}^2$ is known to be backward-complete [23]. (For more details on quasi-metric spaces, we refer to Wilson [26] and Fletcher and Lindgren [12].) By the completeness of the space $(\mathbb{R}^2, d)$, one can apply the Hopf–Rinow Theorem [14] to show the equivalence of the completeness of a metric space and the existence of geodesics in the space. We thus have the following lemma.

Lemma 1 *Let d be a transportation admissible distance on $\mathbb{R}^2$. Then, for any two points $p, q \in \mathbb{R}^2$, there exists a path π from p to q such that the equality $d(p,q) = d(p,r) + d(r,q)$ holds for any point $r \in \pi$.*

Such paths π described above are called *d-straight* (or *geodesics*). Note that d-straight paths generalize straight line segments under the Euclidean metric. This lemma can also be shown by Menger's *Verbindbarkeitssatz* [22, 25]. A similar application of Menger's *Verbindbarkeitssatz* can also be found in Klein and Wood [19].

Immediate examples of transportation admissible distance are the Euclidean metric and the L_1 metric. Since the open metric balls of the L_p metric for $1 \le p \le \infty$ indeed induces the Euclidean topology on $\mathbb{R}^2$ and are bounded by the Euclidean balls, it is easy to see that the L_p metrics are also transportation admissible. By definition, nice metrics in the sense of Klein and Wood [19] are transportation admissible. One remarkable concrete example of nice metrics is the Karlshruhe metric [19], also known as the Moscow metric. Convex distance functions are also transportation admissible distances that may be asymmetric.

2.2 Transportation Networks and Travel Time Distances

Let d be a transportation admissible distance on $\mathbb{R}^2$. Now, we define a transportation network under d, which shall be a plane graph. Note that the transportation networks were defined to be an undirected plane straight-line graph on $\mathbb{R}^2$ when d is either the Euclidean or the L_1 metric in previous work [4, 6]. For a general transportation admissible distance d, we should follow the straightness with respect to d, that is, the d-straightness. In addition, since d may be asymmetric, a transportation network under d should be a directed graph.

Definition 3. Let d be a transportation admissible distance on $\mathbb{R}^2$. A directed, edge-weighted plane graph $G = (V, E, v)$ is called a *transportation network under d* if and only if

(i) each directed edge $e \in E$ is d-straight, and

(ii) the weight $v(e)$ for $e \in E$ is larger than 1.

For a transportation network G, each edge $e \in E$ is called a *road* and its weight $v(e)$ is called the *speed*.

Note that a road of a transportation network has an orientation so that it can be regarded as a one-way road. Two incident nodes of a road $e \in E$ is identified by $p_1(e)$ and $p_2(e)$, where e is directed from $p_1(e)$ to $p_2(e)$.

Let $G = (V, E, v)$ be a transportation network under d. Consider any (directed) path π in $\mathbb{R}^2$ from $p \in \mathbb{R}^2$ to $q \in \mathbb{R}^2$. Then, our speed at any point of path π when traveling along P is determined by the following assumptions of using roads:

1. Along a road $e \in E$, one moves at the specified speed $v(e)(> 1)$ of the road.
2. Out of the roads, one moves at unit speed $(= 1)$.
3. One can access or exit a road at any point on the road.

On the other hand, the distance function d gives us a way to measure the length of any subpath of π. Consequently, we can define the *travel time* of π. For any two points $p, q \in \mathbb{R}^2$, a directed path π from p to q is called a *shortest (travel time) path* in our setting if it minimizes the travel time from p to q. More specifically, if π is piecewise d-straight and thus $\pi = (p = p_0, p_1, \ldots, p_{k-1}, p_k = q)$ as a sequence of breakpoints, then its travel time can be represented as follows:

$$\sum_{i=0}^{k-1} \frac{1}{v_i} d(p_i, p_{i+1}),$$

where $v_i = v(e)$ if $p_i, p_{i+1} \in e$ for some $e \in E$ and p_i precedes p_{i+1} with respect to the orientation of e, or $v_i = 1$, otherwise.

Let $d_G(p, q)$ be the travel time of a shortest path from p to q, and we call d_G the *transportation distance* or *travel time distance induced by d and G*. Such a shortest path always exists due to the existence of d-straight paths. A basic observation is that d_G is also transportation admissible.

Lemma 2 *Let d be a transportation admissible distance and G be a transportation network under d. Then, the travel time distance d_G induced by d and G is also transportation admissible.*

Proof. We verify each of the axioms of transportation admissible distances. Since d is transportation admissible, it is obvious that d_G is a quasi-metric.

To show T2 and T3, it suffices to show that any backward d_G-ball contains a Euclidean ball with the same center and oppositely is contained by a Euclidean ball with the same center. Pick any point $p \in \mathbb{R}^2$. For any $\epsilon > 0$, it is obvious that the backward d_G-ball $B^-_{d_G}(p, \epsilon)$ contains the backward d-ball $B^-_d(p, \epsilon)$. Since d satisfies the axiom T2, $B^-_d(p, \epsilon)$ contains a Euclidean ball centered at p with some radius $\epsilon' > 0$. Now, let v^* be the maximum speed of G, that is, $v^* = \max_{e \in E} v(e)$. Then, $B^-_{d_G}(p, \epsilon)$ is contained in $B^-_d(p, v^*\epsilon)$. Since $B^-_d(p, v^*\epsilon)$ is contained in a Euclidean ball centered at p, so is $B^-_{d_G}(p, \epsilon)$.

Since d_G is induced by shortest paths, it is not difficult to verify axiom T4. Let $p, q \in \mathbb{R}^2$ be any two points and π be a shortest path from p to q. Pick any point $r \in \pi \setminus \{p, q\}$. Then, the subpath of π from p to r is also a shortest path from p to r; otherwise, we get a contradiction to the fact that π is shortest. Similarly, the subpath of π from r to q is also shortest. This implies that $d_G(p, q) = d_G(p, r) + d_G(r, q)$. Thus, axiom T4 is also fulfilled. $\square$

Consequently, by Lemma 1, we can discuss d_G-straight paths, which are in fact shortest travel time paths using the transportation network G. Throughout the paper, we interchangeably use both terms, d_G-straight paths and shortest (travel time) paths.

2.3 Needles

Bae and Chwa [6] defined a needle as a line segment with a weight function on it, which is shown to be useful for analyzing the travel time distance induced by the Euclidean or the L_1 metric and a transportation network.

Here, we generalize the definition of needles in [6] for any transportation admissible distance d.

Definition 4. A *needle* $\mathbf{p}$ is a d-straight path from $p_1(\mathbf{p}) \in \mathbb{R}^2$ to $p_2(\mathbf{p}) \in \mathbb{R}^2$ associated with a *weight function* $w_{\mathbf{p}}$ on $\mathbf{p}$ such that

(i) The weight function $w_{\mathbf{p}}$ is linear
(ii) and it holds that $0 \leq w_{\mathbf{p}}(p_1(\mathbf{p})) \leq w_{\mathbf{p}}(p_2(\mathbf{p}))$.

Also, let $v(\mathbf{p})$ be the reciprocal of the slope of the weight function $w_{\mathbf{p}}$, called the *speed* of needle $\mathbf{P}$. That is, $v(\mathbf{p}) = \frac{d(p_1(\mathbf{p}),p_2(\mathbf{p}))}{w_{\mathbf{p}}(p_2(\mathbf{p}))-w_{\mathbf{p}}(p_1(\mathbf{p}))}$ if $w_{\mathbf{p}}(p_1(\mathbf{p})) < w_{\mathbf{p}}(p_2(\mathbf{p}))$; otherwise, $v(\mathbf{p}) = \infty$.

Note that needles generalize line segments under the Euclidean or L_1 metric with or without a constant weight. Any point or any d-straight path is a special form of a needle under d. We now extend the distance function d from a point to a point to that from a point to a needle. We view the weight $w_{\mathbf{p}}$ of a needle $\mathbf{p}$ as an additive weight so that every point on $\mathbf{p}$ may have an individual additive weight, keeping the linearity along $\mathbf{p}$. In that way, the distance from any point $x \in \mathbb{R}^2$ to a needle $\mathbf{p}$ is defined to be

$$d(x, \mathbf{p}) = \min_{y \in \mathbf{p}} \{d(x, y) + w_{\mathbf{p}}(y)\}.$$

This enables us to define and discuss the Voronoi diagram of needles with respect to any general transportation admissible distance d. For a set S of Voronoi sites, let $\mathcal{V}_d(S)$ be the Voronoi diagram of S with respect to the distance function d. Since we are now able to measure the distance from any point in the plane to a needle under d, the Voronoi diagram $\mathcal{V}_d(\mathcal{S})$ of a set $\mathcal{S}$ of needles is well defined accordingly.

For the Euclidean case, the Voronoi diagram for pairwise non-piercing needles is shown to be an abstract Voronoi diagram in the sense of Klein [16], where two needles are said to be *non-piercing* if and only if the bisector between them contains at most one connected component.

3 Shortest Travel Time Paths and Voronoi Diagrams

In this section, we study the travel time distance function d_G induced by a transportation admissible distance d and a transportation network G. The most fundamental problem about d_G is to evaluate the distance d_G itself and to obtain a corresponding shortest path, that is, a d_G-straight path. We propose an abstract algorithmic scheme that solves the shortest path problem by building a shortest path map, which in turn corresponds to the Voronoi diagram of a certain set of needles. Our method naturally extends to computing the Voronoi diagram with respect to the travel time distance d_G.

3.1 d_G-straight Paths and Needles

As noted in the previous section, a transportation admissible distance d and a transportation network G induce a new distance d_G and d_G-straight paths. The combinatorial structure of any d_G-straight path can be represented by a string of $\{\mathsf{S}, \mathsf{T}\}$, where S represents a maximal d-straight segment without using any road of G and T represents one along a road of G. We

observe that any string that represents a d_G-straight path contains no SS as a substring since an SS can be obviously reduced to an S.

Let us consider a single road e as a simplest case. Given a transportation network G with only one road e, a d_G-straight path is of the form STS or its substring except for SS. This is quite immediate; paths represented by longer strings than STS can be reduced since any road is d-straight by definition and d satisfies the triangle inequality. Thus, any d_G-straight path π from p to q using a road e can be represented as $\pi = (p, p', q', q)$, where p' is the entering point to e and q' is the exiting point to q. We then call q' a *footpoint* of q on e; that is, q' is a footpoint of q on e if there exists a d_G-straight path from any point $p \in \mathbb{R}^2$ to q such that the path uses e and q' is the exiting point. A point may have several or infinitely many footpoints on a road. Let $FP_e(q)$ be the set of footpoints of q on e for all d_G-straight paths from any point to q using e. Now, consider a total order $\prec_e$ on the points on a road e, where $x \prec_e y$ for $x, y \in e$ if x precedes y in the orientation of e. Then, the following property of footpoints can be shown.

Lemma 3 *Let $F \subseteq FP_e(q)$ be a connected component of $FP_e(q)$ for $q \in \mathbb{R}^2$. Then, the set F has the least point q_0 with respect to $\prec_e$ such that $q_0 \in FP_e(q)$ and $d(q_1, q) = d(q_1, q_2)/v(e) + d(q_2, q)$ for any $q_1, q_2 \in F$ with $q_1 \prec_e q_2$.*

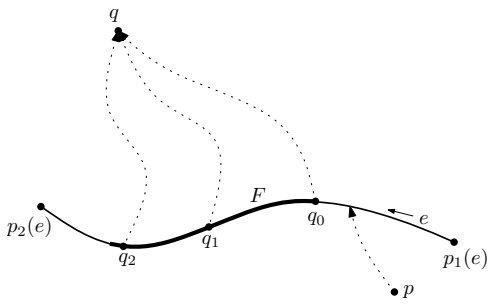


Figure 1: Illustration to the proof of Lemma 3.

Proof. Suppose F is a connected component of $FP_e(q)$ and, in addition, F has the least point q_0. For any $q_1, q_2 \in F$ with $q_1 \prec_e q_2$, if $d(q_1, q) > d(q_1, q_2)/v(e) + d(q_2, q)$, then q_1 is not a footpoint of q since the path $q_1 \to q_2 \to q$ yields a smaller travel time. Let $q_3 \in F$ be the least point such that the equality $d(q_0, q) = d(q_0, q_3)/v(e) + e(q_3, q)$ does not hold. There must exist a shortest path $\pi = (p, p', q_3, q)$ such that $p' \neq q_3$ and $p' \prec_e q_3$. However, if $p' \in F$, $d(p', q) < d(p', q_3)/v(e) + d(q_3, q)$, otherwise, the path $\pi' = (p, p', q_0, q)$ is shorter than π, see Figure 1. Thus, the equality $d(q_1, q) = d(q_1, q_2)/v(e) + d(q_2, q)$ holds for any $q_1, q_2 \in F$.

Now, we show the existence of the least point of F. Let q_0 be the least point of the closure of F with respect to $\prec_e$ and let $q_\epsilon \in F$ such that $d(q_0, q_\epsilon) = \epsilon$ for small $\epsilon > 0$. It is easy to see that $d(q_\epsilon, q)$ is continuous along $\epsilon > 0$. Hence, $d(q_0, q_\epsilon)/v(e) + d(q_\epsilon, q)$ continuously converges to $d(q_0, q)$ as ϵ tends to zero, which implies that q_0 is also a footpoint and is the least point of F. $\square$

For such a shortest path $\pi = (p, p', q', q)$, let $\mathbf{q}$ be a needle on the d-straight subpath of e rom $p_1(\mathbf{q}) = q'$ to $p_2(\mathbf{q}) = p_1(e)$ with weight function $w_{\mathbf{q}}$, where $w_{\mathbf{q}}(q') = d(q', q)$ and $v(\mathbf{q}) = v(e)$. Observe that $d(p, \mathbf{q}) = d_G(p, q)$. Such a needle $\mathbf{q}$ is said to be *produced on a road e from a point q for a footpoint q'*. We let $\sigma_e(q)$ be the set of needles produced on e from q for the least footpoint of every connected component in $FP_e(q)$. Also, both of $FP_e(\cdot)$ and $\sigma_e(\cdot)$ can be naturally extended for needles as done for the distance function d, since shortest paths to a needle under d_G are also d_G-straight paths.

Lemma 4 *If a transportation network G under d consists of a single road e, then for a needle $\mathbf{p}$ and any point $x \in \mathbb{R}^2$, it holds that $d_G(x, \mathbf{p}) = d(x, \sigma_e(\mathbf{p}) \cup \{\mathbf{p}\})$.*

Proof. A needle $\mathbf{q} \in \sigma_e(\mathbf{p})$ implies a piecewise d-straight path of the form STS for a given point x; the path is from x through the closest point on $\mathbf{q}$ and $p_1(\mathbf{q})$ to $\mathbf{p}$ and the length of the path is the same as that to $\mathbf{q}$ from x by definition. And, Lemma 3 tells us that $\sigma_e(\mathbf{p})$ is sufficient to cover all the footpoints $FP_e(\mathbf{p})$ of $\mathbf{p}$. ◻

Now, we consider multiple roads. Let $\sigma_G(\mathbf{p}) := \bigcup_{e \in E} \sigma_e(\mathbf{p})$ and $\sigma_G(A) := \bigcup_{\mathbf{p} \in A} \sigma_G(\mathbf{p})$ for a set A of needles. Since d_G-straight paths may pass through several roads, we apply $\sigma_G(\cdot)$ repeatedly. We thus let $\sigma_G^k(\mathbf{p}) := \sigma_G(\sigma_G^{k-1}(\mathbf{p}))$ and $\sigma_G^0(\mathbf{p}) := \{\mathbf{p}\}$. Also, we let $\mathcal{S}_\mathbf{p}^k$ denote $\bigcup_{i=0}^k \sigma_G^i(\mathbf{p})$ and $\mathcal{S}_\mathbf{p}$ denote $\mathcal{S}_\mathbf{p}^\infty$.

Theorem 1 *Given a transportation admissible distance d and a transportation network G under d, for a point x and a needle $\mathbf{p}$,*

$$d_G(x, \mathbf{p}) = \min_{\mathbf{q} \in \mathcal{S}_\mathbf{p}} d(x, \mathbf{q}).$$

Proof. We first define $d_G^k(p, q)$ be the length of a shortest path from p to q where the path passes through at most k roads in G. Surely, $d_G(p, q) = d_G^\infty(p, q)$.

We claim that $d_G^k(x, \mathbf{p}) = d(x, \mathcal{S}_\mathbf{p}^k)$, which directly implies the theorem. We prove this by induction. Lemma 4 gives us an induction basis. We have

$$d(x, \mathcal{S}_\mathbf{p}^{\ell+1}) = \min\{d(x, \mathcal{S}_\mathbf{p}^\ell), d(x, \sigma_G^{\ell+1}(\mathbf{p}))\} = \min\{d(x, \mathcal{S}_\mathbf{p}^\ell), d(x, \sigma_G(\sigma_G^\ell(\mathbf{p})))\}.$$

By inductive hypothesis and Lemma 4, the equation is evaluated as

$$d(x, \mathcal{S}_\mathbf{p}^{\ell+1}) = \min\{d_G^\ell(x, \mathbf{p}), d_G^1(x, \sigma_G^\ell(\mathbf{p}))\}.$$

As pointed out in the proof of Lemma 4, $d_G^1(x, \sigma_G^\ell(\mathbf{p}))$ implies a shortest path to a needle in $\sigma_G^\ell(\mathbf{p})$ using exactly one road in G, and further a shortest path to $\mathbf{p}$ using exactly $\ell + 1$ roads. Therefore, we conclude $d(x, \mathcal{S}_\mathbf{p}^{\ell+1}) = d_G^{\ell+1}(x, \mathbf{p})$, implying the theorem. ◻

Theorem 1 describes a nice relation between needles and roads. For a given point $p \in \mathbb{R}^2$, a *shortest path map* for p is a subdivision of $\mathbb{R}^2$ into regions according to the combinatorial structure of d_G-straight paths to p. Theorem 1 gives us that the travel time distance d_G to $p \in \mathbb{R}^2$ can be evaluated by taking the minimum distance to $\mathbf{q}$ over $\mathbf{q} \in \mathcal{S}_p$, and this can be done by maintaining the Voronoi diagram $\mathcal{V}_d(\mathcal{S}_p)$ of the set $\mathcal{S}_p$ of needles. Moreover, the Voronoi diagram plays a role as a shortest path map.

Corollary 1 *For a given $p \in \mathbb{R}^2$, the Voronoi diagram $\mathcal{V}_d(\mathcal{S}_p)$ of the set of produced needles for p coincides with a shortest path map for p.*

The Voronoi diagram under the travel time metric d_G can be viewed as the case of multiple sources. For a set S of finite points in $\mathbb{R}^2$, let $\mathcal{S}$ be the union $\bigcup_{p \in S} \mathcal{S}_p$. Then, Theorem 1 implies that the Voronoi diagram $\mathcal{V}_d(\mathcal{S})$ under d induces the Voronoi diagram $\mathcal{V}_{d_G}(S)$ of S under the travel time metric d_G. In other words, any Voronoi region in $\mathcal{V}_d(\mathcal{S})$ is completely contained in a Voronoi region in $\mathcal{V}_{d_G}(S)$, that is, $\mathcal{V}_{d_G}(S)$ is a sub-diagram of $\mathcal{V}_d(\mathcal{S})$.

Corollary 2 *For any set S of points in $\mathbb{R}^2$, the Voronoi diagram $\mathcal{V}_{d_G}(S)$ of S under the travel time distance d_G can be extracted from $\mathcal{V}_d(\mathcal{S})$ in time linear to the size of $\mathcal{V}_d(\mathcal{S})$.*

Hence, we have a algorithmic framework that computes the Voronoi diagram $\mathcal{V}_{d_G}(S)$ of a given set S of sites with respect to the travel time distance d_G with three phases. First, we

compute the set $\mathcal{S} = \bigcup_{s \in S} \mathcal{S}_s$ of needles. Second, we compute the Voronoi diagram $\mathcal{V}_d(\mathcal{S})$, and then the target diagram $\mathcal{V}_{d_G}(S)$ is finally obtained from $\mathcal{V}_d(\mathcal{S})$ by Corollary 2. Note that in order to have a shortest path map for a point $s \in \mathbb{R}^2$, perform the first and second phases with $S = \{s\}$, resulting in $\mathcal{V}_d(\mathcal{S}_s)$, which is a shortest path map for s by Corollary 1.

The second phase, computing $\mathcal{V}_d(\mathcal{S})$, would be solved by several technics and general approaches to compute Voronoi diagrams, such as the abstract Voronoi diagram [16]. Here, we focus on how to compute the set $\mathcal{S}$ of needles—in fact, a finite subset of $\mathcal{S}$ that is equivalently effective—in an efficient way, keeping this abstraction level at the moment.

3.2 Computing Effective Needles

Recall that $\mathcal{S}$ is defined as all the needles recursively produced from given sites, and may contain infinitely many needles that have an empty Voronoi region in $\mathcal{V}_d(\mathcal{S})$. We call a needle $\mathbf{p} \in \mathcal{S}$ *effective with respect to* $\mathcal{S}$ if the Voronoi region of $\mathbf{p}$ in $\mathcal{V}_d(\mathcal{S})$ is not empty. Let $\mathcal{S}^* \subseteq \mathcal{S}$ be the maximal set of effective needles with respect to $\mathcal{S}$, that is, we have $\mathcal{V}_d(\mathcal{S}^*) = \mathcal{V}_d(\mathcal{S})$. We describe a high-level algorithm that computes $\mathcal{S}^*$ from G and S.

The algorithm works with handling events, which are defined by a certain situation at a time. Here, at each time t, we implicitly maintain the (backward) d_G-balls of the given sites, where the backward d_G-ball of a site p is defined as the set $B^-_{d_G}(p, t) = \{x \in \mathbb{R}^2 \mid d_G(x, p) < t\}$, and d_G-balls expand as time t increases from $t = 0$. Here, we have only one kind of events, called *birth events* that occur when a d_G-ball touches any footpoint on a road during their expansions; at a birth event, a new needle will be produced in the algorithm. We can determine a birth event associated with a footpoint of a needle on a road. In order to handle events, we need two data structures:

(i) Let $\mathcal{Q}$ be an event queue implemented as a priority queue such that the priority of an event e is its occurring time and $\mathcal{Q}$ supports inserting, deleting, and extracting-minimum in logarithmic time with linear space.

(ii) Let $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_m$ be balanced binary search trees, each associated with e_i, where the road set E is given as $\{e_1, e_2, \ldots, e_m\}$. Each $\mathcal{T}_i$ stores needles on e_i in order and the precedence for a needle $\mathbf{p}$ follows from that of $p_1(\mathbf{p})$ with respect to $\prec_{e_i}$. $\mathcal{T}_i$ supports inserting and deleting of a needle in logarithmic time, and also a linear scan for needles currently in $\mathcal{T}_i$ in linear time and space.

Now, we are ready to describe the algorithm ComputeEffectiveNeedles. First, the algorithm computes $\sigma_G(S)$ and the associated birth events, and insert events into $\mathcal{Q}$. Then, while the event queue $\mathcal{Q}$ is not empty, repeat the following procedure: (1) Extract the next upcoming event b, say that b is a birth event on a road e_i associated with a needle $\mathbf{p}$, (2) test the effectiveness of $\mathbf{p}$, and (3) if the test has passed, compute birth events associated with $\sigma_G(\mathbf{p})$, and insert the events into $\mathcal{Q}$.

Algorithm ComputeEffectiveNeedles returns exactly $\mathcal{S}^*$ by the effectiveness test in step (2). This test can be done by checking if the associated footpoint of the current event has been already dominated by the d_G-balls of the other sites. Thus, if the test is passed, then the new needle is effective and thus we insert it into $\mathcal{T}_i$. The following lemma shows that the effectiveness test is necessary and sufficient to compute $\mathcal{S}^*$.

Lemma 5 *For every birth event and its associated needle* $\mathbf{p}$, $\mathbf{p}$ *is effective with respect to* $\mathcal{S}$ *if and only if it passes the effectiveness test of Algorithm* ComputeEffectiveNeedles.

Proof. First, assume $\mathbf{p}$ has passed the effectiveness test. Then, at time $t = w_{\mathbf{p}}(p_1(\mathbf{p}))$, no d_G-balls would dominate the point $p_1(\mathbf{p})$, but $\mathbf{p}$ does. That is, the region of $\mathbf{p}$ is not empty. Thus, $\mathbf{p}$ is effective.

Input. A transportation network G and a set S of sites under d
Output. The set $\mathcal{S}^*$ of effective needles such that $\mathcal{V}_d(\mathcal{S}^*) = \mathcal{V}_d(\mathcal{S})$.

```
Compute σ_G(S) and the associated birth events, and insert events into Q.
while Q is not empty do
    Extract the next upcoming event b.
    if b is a birth event associated with p on e_i then
        Test the effectiveness of p.
        If p is effective, then insert it into T_i.
        Compute σ_G(p) and the associated birth events, and insert events into Q.
    end if
end while
Report S ∪ ⋃_{1≤i≤m} T_i as S*.
```

Figure 2: Algorithm ComputeEffectiveNeedles

If $\mathbf{p}$ is effective and $p_1(\mathbf{p})$ does not belong to the region of $\mathbf{p}$, neither do other points on $\mathbf{p}$ since $\mathbf{p}$ is on a road and its speed is the same as that of the road. Note that if $\mathbf{p}$ dominate no points on $\mathbf{p}$, the Voronoi region of $\mathbf{p}$ is empty. Consequently, the region of $\mathbf{p}$ should be empty, which leads to a contradiction. □

The algorithm always terminates, which can be shown by the finiteness of $\mathcal{S}^*$.

Lemma 6 *$\mathcal{S}^*$ is finite and the number of handled events is $O(s \cdot |\mathcal{S}^*|)$, where s is the maximum cardinality of $\sigma_G(\mathbf{p})$ for any needle $\mathbf{p}$.*

Proof. The effectiveness test always fails if all the roads are covered by the d_G-balls after sufficiently large time since the d-distances between any two points in the plane are bounded. Hence, the set $\mathcal{S}^*$ consists of a finite number of needles. Since any needle $\mathbf{p}$ produces $O(s)$ birth events, only when it passes the effectiveness test. the number of events handled while running the algorithm is also bounded by $O(s \cdot |\mathcal{S}^*|)$. □

We end this section with the following conclusion.

Theorem 2 *Given a transportation admissible distance d, a transportation network G under d, and a set S of sites, $\mathcal{S}^*$ can be computed in $O(s \cdot |\mathcal{S}^*| \cdot (\log(s \cdot |\mathcal{S}^*|) + T_{eff}))$ time, where where s is the maximum cardinality of $\sigma_G(\mathbf{p})$ for any needle $\mathbf{p}$, and T_{eff} denotes the maximal time taken to perform an effectiveness test.*

4 Transportation Networks under Convex Distances

In this section, we consider convex distances as a concrete example of transportation admissible distances. We thus investigate geometric and algorithmic properties of travel time distances induced by a convex distance and a transportation network, and obtain efficient algorithms that compute the shortest path map or the Voronoi diagram with respect to the travel time distance.

In order to devise such an algorithm, we apply our algorithmic framework described in the previous section. For the purpose, a bundle of algorithmic modules have to be precisely described: how to compute needles produced from a needle, how to test the effectiveness of needles, how many needles and events to handle, how to compute the Voronoi diagram for

needles, and some technical lemmas to reduce the complexity.

A *convex distance* is defined by a compact and convex body C containing the origin, or the *reference point*, and is measured as the factor that C centered at the source should be expanded or contracted for its boundary to touch the destination. Note that a convex distance is symmetric, i.e. being a metric, if and only if C is centrally symmetric at its reference point.

We consider the convex body C as a black box which supports several elementary operations. These include finding the Euclidean distance from the reference point to the boundary in a given direction, finding two lines which meet at a given point and are tangent to C, finding the footpoint for a needle and a road, and computing the bisecting curve between two sites under the convex distance based on C. Here, we assume that these operations can be performed in a reasonable time bound.

Throughout this section, for a convex body $C \subset \mathbb{R}^2$, we denote by $C + p$ a translate of C by a vector $p \in \mathbb{R}^2$ and by λC a scaled copy of C by a factor $\lambda \geq 0$. And, we may denote by C' the reflected body of C with respect to the origin and by ∂C the boundary of C.

4.1 Roads and needles under a convex distance

Let d be a convex distance function on $\mathbb{R}^2$ based on a convex body C with a reference point. Note that any line segment is a d-straight path by the convexity.

For the Euclidean distance, to take a shortest path with a road e, one should enter or exit e with angle $\pi/2 \pm \alpha$, where $\sin\alpha = 1/v(e)$ [2]. For a convex distance d based on C, there also exist such entering or exiting angles for a road that they lead to a shortest path. We can obtain these angles by simple operations on C. First, we pick a point x on e not $p_2(e)$ and consider the set $(d(x, p_2(e))/v(e))C' + p_2(e)$, denoted by D. We then find two lines that meet at x and are tangent to D. Each of the two lines is above or below e. We let α_e^+ and α_e^- be two inward directions perpendicular to the two lines. Then, consider two directions from $p_2(e)$ to two meeting points between the two lines and D. We then let β_e^+ and β_e^- be reflections of the two directions at the reference point of D. The symbols $+$ and $-$ mean "above" and "below" with respect to e, respectively. See Figure 3(a).

Lemma 7 *Given a road e under a convex distance d, any shortest path from p to q passing through e has the following properties:*

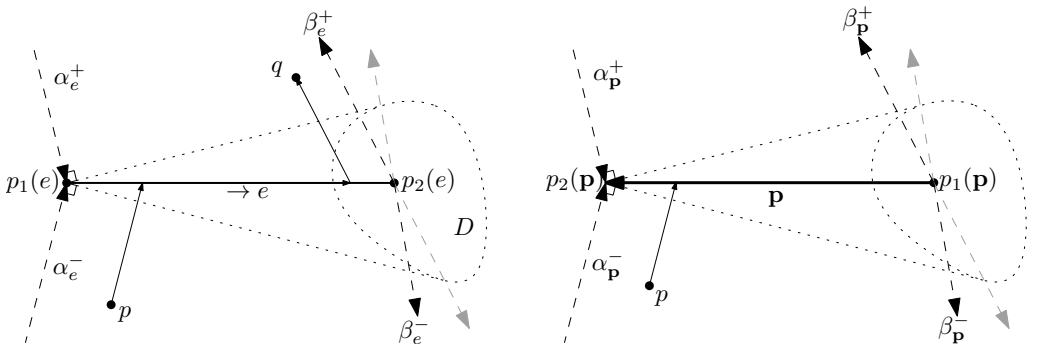
- *If p is above e, either the access direction is α_e^+ or the access point is $p_1(e)$.*
- *If p is below e, either the access direction is α_e^- or the access point is $p_1(e)$.*
- *If q is above e, either the leaving direction is β_e^+ or the exiting point is $p_2(e)$.*
- *If q is below e, either the leaving direction is β_e^- or the exiting point is $p_2(e)$.*

By Lemma 7, we can easily find a shortest path with one road. For instance, Figure 3(a) shows a shortest path from p to q using road e.

We can define similar terms for a needle as we did for a road. The (backward) d-ball $B_d^-(\mathbf{p}, t)$ of $\mathbf{p}$ is described as follows:

- When $0 < t \leq w_{\mathbf{p}}(p_1(\mathbf{p}))$, $B_d^-(\mathbf{p}, t)$ is an empty set.
- When $w_{\mathbf{p}}(p_1(\mathbf{p})) < t \leq w_{\mathbf{p}}(p_2(\mathbf{p}))$, $B_d^-(\mathbf{p}, t)$ is the convex hull of $(t - w_{\mathbf{p}}(p_1(\mathbf{p})))C' + p_1(\mathbf{p})$ and the point x on $\mathbf{p}$ such that $d(x, p_1(\mathbf{p}))/v(\mathbf{p}) = w_{\mathbf{p}}(x) - w_{\mathbf{p}}(p_1(\mathbf{p}))$.
- When $t > w_{\mathbf{p}}(p_2(\mathbf{p}))$, $B_d^-(\mathbf{p}, t)$ is the convex hull of $(t - w_{\mathbf{p}}(p_1(\mathbf{p})))C' + p_1(\mathbf{p})$ and $(t - w_{\mathbf{p}}(p_2(\mathbf{p})))C' + p_2(\mathbf{p})$.

Thus, when $t > w_{\mathbf{p}}(p_2(\mathbf{p}))$, $\partial B_d^-(\mathbf{p}, t)$ contains two line segments tangent to two scaled copies of C' and the slopes of these line segments do not change while t changes. Hence, the meeting points between the line segments and the convex bodies scaled from C' make four rays, and



(a) A road e and a shortest path from p to q (b) A needle $\mathbf{p}$ and a shortest path from p to $\mathbf{p}$

Figure 3: Directions defined for a road and for a needle

they have two directions $-\beta_{\mathbf{p}}^+$ and $-\beta_{\mathbf{p}}^-$. Then, we can define $\alpha_{\mathbf{p}}^+$, $\alpha_{\mathbf{p}}^-$, $\beta_{\mathbf{p}}^+$, and $\beta_{\mathbf{p}}^-$, equivalently as for a road, see Figure 3(b). Note that if $\mathbf{p} \in \sigma_e(\mathbf{q})$ for any needle $\mathbf{q}$, then $\alpha_{\mathbf{p}}^+ = \alpha_e^+$, $\alpha_{\mathbf{p}}^- = \alpha_e^-$, $\beta_{\mathbf{p}}^+ = \beta_e^+$, and $\beta_{\mathbf{p}}^- = \beta_e^-$, by definition of $\sigma_e(\mathbf{q})$.

4.2 Computing the effective set of needles

In this subsection, we discuss how to compute the set $\mathcal{S}^*$ of effective needles for given sites S. We follow the abstract algorithm ComputeEffectiveNeedles described in the previous section, and fill the details in each subroutine.

The footpoints. Under a convex distance, any needle has at most one connected component of footpoints on a road. Lemma 7 tells us how to compute the least footpoint of a point p on a road e; it can be obtained as either the intersecting point of the road e and the ray with direction $-\beta_e^+$ or $-\beta_e^-$ from p, or just $p_2(e)$.

Lemma 8 *Let d be a convex distance based on a convex C and G be a transportation network under d. For a road e in G and a needle $\mathbf{p}$, the number of connected components of footpoints of $\mathbf{p}$ on e is at most one and the least footpoint is either the least footpoint of $p_1(\mathbf{p})$ or $p_2(\mathbf{p})$, or just $p_2(e)$.*

Proof. First, we let $\psi_{\mathbf{p}}(x)$ be the nearest point of $\mathbf{p}$ from a point x such that $d(x, \mathbf{p}) = d(x, \psi_{\mathbf{p}}(x)) + w_{\mathbf{p}}(x)$. Note that a needle $\mathbf{p}$ can be viewed as weighted points on a line segment whose weight are linearly assigned along the segment. Therefore, if a point x on a road e, not an endpoint of $\mathbf{p}$, is a footpoint and $\psi_{\mathbf{p}}(x)$ is not an endpoint of $\mathbf{p}$, we can force the footpoint x to be $p_1(e)$ or $\psi_{\mathbf{p}}(x)$ to be an endpoint of $\mathbf{p}$ since x is not a footpoint of $\psi_{\mathbf{p}}(x)$ on e. $\square$

The effectiveness test. At every time a birth event occurs, the effectiveness test is done by testing if the point where the event occurs is dominated by others that are already produced and effective at that time. Under a convex distance d, d-balls of any needle are convex so that we can test the effectiveness in logarithmic time by maintaining the tree structures $\mathcal{T}_i$. We refer to [6] for more details about the effectiveness test.

The number of needles and events. By convexity, we can show a couple of technical lemmas that prove the number of needles and events we handle.

Lemma 9 *Let* $\mathbf{p}$ *be a needle produced on a road* e *from a needle* $\mathbf{q}$*. For another road* $e' \in E$*, if* $\mathbf{p}$ *does not dominate any node of* e *or* e'*, no needles in* $\sigma_{e'}(\mathbf{p})$ *are effective with respect to* $\mathcal{S}$*.*

Proof. By Lemma 8, candidates of the footpoint of $\mathbf{p}$ on e' are a node of e' and the two footpoints of two endpoints of $\mathbf{p}$ on e'. Hence, if $\mathbf{p}$ does not dominate any node of e, there should exist another effective needle dominating $p_2(\mathbf{p})$ and the footpoint of $p_1(\mathbf{p})$ should be dominated by $\mathbf{q}$. Further, if $\mathbf{p}$ dominates no nodes of e', all the candidates to produce effective needles from $\mathbf{p}$ are dominated by other ones. Thus, the lemma is shown. ◻

Lemma 10 *The set* $\mathcal{S}^*$ *consists of at most* $O(m(n+m))$ *needles, where* n *is the number of sites in* S *and* m *is the number of roads in* G*. In addition, the number of handled events while the algorithm* ComputeEffectiveNeedles *running is* $O(m^2(n+m))$*.*

Proof. By Lemma 8, for any needle $\mathbf{p}$, $\sigma_G(\mathbf{p})$ contains only $O(m)$ needles. And, by Lemma 9, only needles dominating any node may produce other effective needles. Consequently, we have $O(m(n+m))$ effective needles in $\mathcal{S}$. Moreover, by Lemma 6, the number of events we handle while the algorithm is executed is $O(m^2(n+m))$. ◻

Improvement via primitive paths. Consequently, we have an algorithm to compute $\mathcal{S}^*$ in $O(m^2(n+m)(\log(n+m)+T_{op}(C)))$ time with $O(m^2(n+m))$ space by Theorem 2, where $T_{op}(C)$ is time taken to perform a basic operation on C. This complexity, however, can be reduced by small modifications on the algorithm. For the Euclidean case, Bae and Chwa [6] additionally maintains *node events*, which occur when $B^-_{d_G}(p,t)$ first touches a node, to reduce the number of events to $O(m(n+m))$. Here, we also can apply the approach by Lemma 9.

The authors also introduced primitive paths; a path is called *primitive* if the path contains no nodes in its interior and passes through at most one road. We can show the lemma of primitive paths in our setting, following from Lemma 9.

Lemma 11 *Given a transportation network* $G = (V, E, v)$ *under a convex distance* d*, for two points* p *and* q*, there exists a* d_G*-straight path* π *from* p *to* q *such that* π *is a sequence of shortest primitive paths whose endpoints are* p*,* q*, or nodes in* V*.*

The proof of Lemma 11 is almost identical to Lemma 7 in Bae and Chwa [6], and thus we omit its proof. By Lemma 11 together with node events, we can improve our algorithm. Indeed, we can compute a shortest path for two given points by constructing an edge-weighted complete graph such that vertices are nodes and two given points, and edges are shortest primitive paths among vertices. Furthermore, we can use the graph to avoid useless computations during the algorithm. For more details, we refer to [6].

Lemma 12 *One can compute* $\mathcal{S}^*$ *in* $O(m(n+m)(m+\log n+T_{op}(C)))$ *time with* $O(m(n+m))$ *space.*

4.3 Voronoi Diagrams for Needles

In general, bisectors between two needles under a convex distance can be parted into two connected components. (See [6] for more details.) However, it will be shown that $\mathcal{S}^*$ can be replaced by such nice needles, so called *non-piercing*, that the Voronoi diagram for them is an abstract Voronoi diagram which can be computed in the optimal time and space.

Computing the Voronoi diagrams for non-piercing needles. The *abstract Voronoi diagram* is a unifying approach to define and compute general Voronoi diagrams, introduced by Klein [16]. In this model, we deal with not a concrete distance function but the family of

bisecting curves $J(p,q)$ defined in an abstract fashion between two sites p and q. A bisecting curve $J(p,q)$ is supposed to partition the plane $\mathbb{R}^2 \setminus J(p,q)$ into two regions, $R(p,q)$ and $R(q,p)$. A system $(S, \{J(p,q) \mid p,q \in S, p \neq q\})$ of bisecting curves for S is called *admissible* if the following conditions are fulfilled:

A1 $J(p,q)$ is homeomorphic to a line or empty,
A2 $R(p,q) \cap R(q,r) \subset R(p,r)$,
A3 for any subset $S' \subseteq S$ and $p \in S'$, the set $R(p,S') := \bigcap_{q \in S' \setminus \{p\}} R(p,q)$ is path-connected, and
A4 the intersection of any two bisectors consists of finitely many components.

In fact, the first three conditions are enough to handle abstract Voronoi diagrams theoretically but the fourth one is necessary in a technical sense [17]. Though all convex distances satisfy the first three ones, there exist convex distances violating the fourth one. We guarantee the fourth condition by postulating that ∂C is semialgebraic [10]. We also note that two-dimensional bisectors can be avoided by a total order on given sites [19].

Let $\mathcal{S}$ be any finite set of needles under d. For any $\mathbf{p}, \mathbf{q} \in \mathcal{S}$ we set $J(\mathbf{p},\mathbf{q}) = \{x \in \mathbb{R}^2 \mid d(x,\mathbf{p}) = d(x,\mathbf{q})\}$ and $R(\mathbf{p},\mathbf{q}) = \{x \in \mathbb{R}^2 \mid d(x,\mathbf{p}) < d(x,\mathbf{q})\}$. We then have a system $(\mathcal{S}, \{J(\mathbf{p},\mathbf{q}) \mid \mathbf{p},\mathbf{q} \in \mathcal{S}, \mathbf{p} \neq \mathbf{q}\})$ of bisecting curves. Two needles are called *non-piercing* if the bisecting curve between them is connected. The following lemma shows that the non-piercing condition is nice enough to restrict a bisector system for needles to be admissible.

Lemma 13 *Let $\mathcal{S}$ be a set of pairwise non-piercing needles under a convex distance d based on C whose boundary is semialgebraic. Then, the system $(\mathcal{S}, \{J(\mathbf{p},\mathbf{q}) \mid \mathbf{p},\mathbf{q} \in \mathcal{S}, \mathbf{p} \neq \mathbf{q}\})$ of bisecting curves for $\mathcal{S}$ is admissible.*

Proof. Condition A4 immediately follows from the assumption that ∂C is semialgebraic. The bisecting curve $J(\mathbf{p},\mathbf{q})$ is homeomorphic to a line due to the convexity of the d-ball of needles. Thus, condition A1 is fulfilled.

For any $x \in \mathbb{R}^2$ and $\mathbf{p} \in \mathcal{S}$, let $\psi_{\mathbf{p}}(x)$ be defined as in the proof of Lemma 8. Observe that if $x \in R(\mathbf{p},\mathbf{q})$, then any point in the line segment between x and $\psi_{\mathbf{p}}(x)$ lies in $R(\mathbf{p},\mathbf{q})$ since the line segment is d-straight. This implies that $R(\mathbf{p},\mathbf{q})$ is path-connected, and so is $R(\mathbf{p},\mathcal{S}')$ for any subset $\mathcal{S}' \subseteq \mathcal{S}$. Hence, condition A3 is shown.

In order to see condition A2, pick any point $x \in R(\mathbf{p},\mathbf{q}) \cap R(\mathbf{q},\mathbf{r})$. If there is no such point, that is, the intersection is empty, A2 is trivially true. We then have $d(x,\mathbf{p}) < d(x,\mathbf{q}) < d(x,\mathbf{r})$, which implies that $x \in R(\mathbf{p},\mathbf{r})$. □

There are several optimal algorithms computing abstract Voronoi diagrams [11, 16, 18, 21]. These algorithms assume that the bisector between two sites can be computed in constant time, and construct the Voronoi diagram in $O(n \log n)$ time with $O(n)$ space when n sites are given. In our setting, we assume that a representation of the bisector between two needles has the complexity $S_b(C)$, and can be computed in $T_b(C)$ time.

Corollary 3 *Let $\mathcal{S}$ be a finite set of pairwise non-piercing needles under a convex distance d based on a convex body C whose boundary is semialgebraic. Then, the Voronoi diagram $\mathcal{V}_d(\mathcal{S})$ can be computed in $O(T_b(C) \cdot |\mathcal{S}| \log |\mathcal{S}|)$ time and $O(S_b(C) \cdot |\mathcal{S}|)$ space.*

Making $\mathcal{S}^*$ pairwise non-piercing. We can make $\mathcal{S}^*$ pairwise non-piercing by the procedure introduced in the proof of Lemma 2 in [6]. The only difference arises when we consider non-piercing needles as input sites; the produced needles on the roads may pierce the original needles. This problem can be solved by cutting a pierced original needle into two non-pierced needles. Since these piercing cases can occur only between original needles and produced needles

dominating a node, we can check all the cases in $O(T_{op}(C) \cdot mn)$ time and the asymptotic number of needles does not increase. We denote by $\mathcal{S}^*_{np}$ the resulting set of pairwise non-piercing needles for a given set S of pairwise non-piercing needles. Note that $\mathcal{V}_d(\mathcal{S}^*_{np})$ is a refined diagram of $\mathcal{V}_d(\mathcal{S}^*)$.

4.4 Putting It All Together

From the previous discussions, we finally conclude the following theorem.

Theorem 3 *Let d be a convex distance based on a convex C whose boundary is semialgebraic, G be a transportation network with m roads under d, and S be a set of n sites. Then, the Voronoi diagram $\mathcal{V}_{d_G}(S)$ under d_G can be computed in $O(m(n+m)(m + T_b(C)\log(n+m) + T_{op}(C)))$ time with $O(S_b(C)m(n+m))$ space, where $T_{op}(C)$, $T_b(C)$, and $S_b(C)$ are defined as before.*

If C is a k-gon, then we have $T_{op}(C) = O(\log k)$ and $T_b(C) = S_b(C) = O(k)$ [20].

Corollary 4 *Let d be a convex distance based on a convex k-gon C, G be a transportation network with m roads under d, and S be a set of n sites. Then, the Voronoi diagram $\mathcal{V}_{d_G}(S)$ under d_G can be computed in $O(m(n+m)(m + k\log(n+m)))$ time with $O(km(n+m))$ space.*

5 Concluding Remarks

We have presented an abstract definition of a class of distance functions that can be facilitated by a transportation network. Under a transportation admissible distance, transportation networks are well defined and we can discuss shortest travel time paths and the travel time distance. To analyze the travel time distance, needles are shown to provide a systematic tool in this general setting. We have shown that there is a finite set of needles for given sites such that the travel time distance to each site can be described by the Voronoi diagram of the set of needles.

Nonetheless, transportation admissible distances are so general that it would be difficult to efficiently compute shortest travel time paths and Voronoi diagrams for hard instances. A research direction would be thus to finding a subclass of transportation admissible distances that admit efficient algorithms for the problems.

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