Outstanding Principal as Prepayment Value: A Closed-Form Formula for Mortgage Pricing

Yi-Cheng Tsai, Chin-Laung Lei, Jan-Ming Ho, Ming-Yang Kao, and Szu-Lang Liao

Abstract— **Mortgage is one of the most popular instruments in the financial markets. In this study, we consider three actions, i.e., to default, to prepay, and to maintain the mortgage, that a borrower may have in mortgage horizon. We provide an effective pricing formula, which not only considers the effect that default might affect the mortgage value, but also accurately computes the impact due to prepayment risk. Our model defines the prepayment value of the mortgage as the amount of outstanding principal, in contrast to defining prepayment value as a constant proportion of maintaining value of the mortgage. We present a new closed-form pricing formula of risky mortgage and also derive its yield to maturity, duration and convexity to provide a framework for risk management.**

Keywords: **Mortgage, Prepayment, Default, Yield to maturity, Duration, Convexity, Risk management.**

I. Introduction

DURING the world economic crises in 2007-2009, banks and financial institutions around the world have suffered huge losses. One of the major source of these crises came from the meltdown of the mortgage market which is the biggest segment of the U.S. fixed income markets [1]. In the second quarter of 2008, the total value of mortgages outstanding in the U.S. is $\$11.3$ trillion and the total value of mortgage securities outstanding is $\$6.8$ trillion [2]. To value a mortgage and understand the underlying risk is an essential step towards maintaining stability of the mortgage market.

Fixed-rate mortgage, denoted as FRM, is a fully amortizing mortgage loan with fixed monthly payments and interest rate through the term of the loan. In the ideal condition, i.e., borrowers maintain their mortgages till the maturity date and the risk-free interest rate is fixed through the contract, we can obtain mortgages values by discounting fixed monthly cash flow. However, it is not the case in reality. The risk-free interest rate changes with time, and borrowers may prepay or default their debts. The unpredictable payment behavior of a borrower under the fluctuations of economic environment is a risk of banks and investment managers. So, in this paper, we'll present a model and its solution of pricing mortgages effectively and accurately with consideration of payment behavior of a borrower as risk factors.

Valuation of risky mortgages has been studied in previous literatures. These methods can be grouped into two types based on how to model the probabilities of termination risks. The first category of these methods are structural-form models, also called endogenous or option-based models. This kind of models originally come from credit risk models developed by Merton [3]. Kau and Keenam [4] gave a survey of related literatures up to 1995. The basic idea is to optimize borrowers' profits by deciding to continue the mortgage, to prepay or to default. They treat prepayment as American-style call options, because borrowers have the rights to prepay at anytime to maximize their profits. Default can be consider as European compound put option. In a default event, the borrower turns over possession of the house and stops paying the mortgage. However, there are many literatures(see Collin-Dufresne and Harding [5]; Kau et al. [6]; Altman [7]) pointing out the drawbacks of the structure-form models including the complicated numerical calculation involved to solve the partial differential equations.

Another category of methods named reduced-form model, also called exogenous or intensity-based model, has been used in mortgage pricing. The main idea is to model the state variable (e.g., interest rate) as a stochastic process, and use the state variables to describe the borrowers' behaviors. The termination risks are assumed to be Poisson distributions, and the hazard rates, i.e., intensities of the Poisson distributions, are described by state variables. Because reduced-form models do not involve complicated numerical calculation to solve the partial differential equations of American-style options, several works use this kind of model to price risky mortgage. Kau et al. [6] presented a FRM valuation model using the reduced-form model, and they use Monte Carlo simulation to derive mortgage price. Pliska [8], and Longstaff [9] developed a multi-stage decision model based on the reduced-form model which considers prepayment risk. Gorovoy and VadimLinetsky [1] present a closed-form solution that models prepayment risk. Tsai et al. [10] derived a closed-form solution for calculating mortgage value under continuous-time intensity-form model. They assumed that intensity rates of prepayment and default are linear functions of interest rate. Their model includes jump process to model the occurrence of non-financial events that cause prepayment

An earlier version of this paper was presented at the Computational Intelligence for Financial Engineering and Economics (CIFEr), 2012 IEEE Conference, New York, 29-30 March 2012.

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and default. They also assumed the loss rate of mortgage value to be a constant regardless of the timing when the borrower prepays the mortgage.

In this paper, to account for termination risks, we derive a closed-form formula for valuing of FRM. Our evaluation model is based on the continuous-time reduced-form model. We use backward recursion method to capture the possible behaviors of borrower. Following Duffie et al. [11], we assumed that the intensity rates of prepayment and default are proportional to the interest rate (the state variable in our economy). To take into account the financial events (triggers), the jump structure is also included. In literature [12], there are several models of interest rate process. For example, Kou [13] proposed the interest rate process follows a double exponential jump diffusion process. This model incorporates both mean reversion and jump diffusion, and it also considers the empirical phenomenon called "volatility smile" in interest-rate markets. We use this model to express interest rate process.

The major contribution of this paper is that we give an exact model of the impact of prepayment risk in the pricing model. Tsai et al. [10] assumed when prepayment or default occurs, the lender will receive partial amount of money which is proportional to the value when the borrower maintains this contract. This ratio of the amount of money received to the value of the mortgage is called recovery rate, and is assumed to be constant in their paper. For a default event, the amount of money lender might recover from auctioning the estate is usually a constant proportion of the value of the mortgage. Jokivuolle and Peura [14] show that it makes no significant difference on pricing mortgages to define the default recovery rate as either a constant or a random variable. However, the amount of money that the lender receives in prepayment event equals to outstanding principal. The outstanding principal of FRMs on each payment day can be calculated. Using such recovery rate to estimate the amount of money that lender will receive in prepayment event will induce additional error. We can take the following simple example to check if the prepayment loss rate, which is one minus recovery rate, will not change with time. Consider a 20-year monthly payment mortgage loan with 1 million initial outstanding and 0.05 interest rate. Because this mortgage is a FRM, the payment on each payment day is about $6,600$. By assuming the risk free interest rate is always 0.02, then we can obtain the initial value of this mortgage to be approximately $1,304,561$ by discounting cash flow on each payment day. By definition, the prepayment loss rate is

$$1 - (prepaymentvalue/mortgagevalue). \tag{1}$$

The prepayment value is outstanding principal. Then we can derive the prepayment loss rate as

$$1 - (outstandingprincipal/mortgagevalue). \tag{2}$$

We compute the prepayment loss rate on each payment day and plot as Fig. 1. From Fig. 1, the prepayment loss rate decreases from 0.23 to 0.002. It is obvious that the prepayment loss rate is different with time.

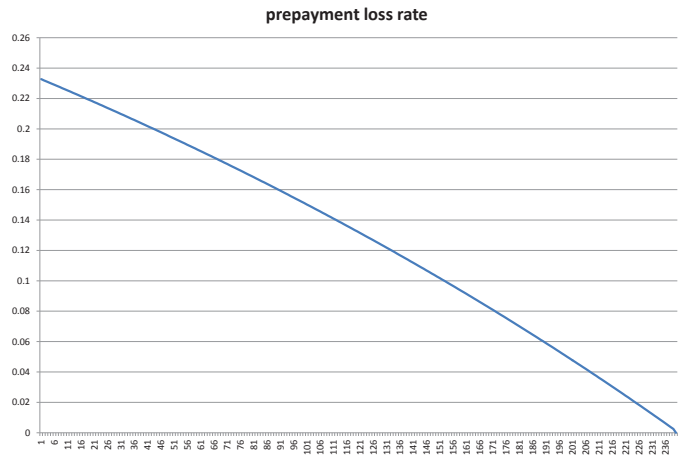

Fig. 1. prepayment loss rate on each payment day

The mortgage value with prepayment risk is affected significantly by the difference between risk-free interest rate and coupon rate of the mortgage. Fig.2 shows one month Libor rate from September, 1991 to September, 2011. The Libor rate is 5.5% percentages in September, 1991 and 0.23% in September, 2011. A FRM is a 20 years contract from September, 1990 to September, 2010. The value of the mortgage to the lender is different for the borrower to prepay at the two different points in time. Our method can compute mortgages precisely under such dynamics. We also derive a closed-form formula for the computation of mortgage value.

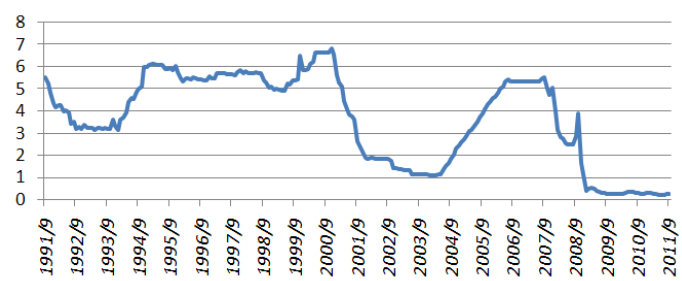

Fig. 2. One month Libor rate from September, 1991 to September, 2011. Data from http://mortgage-x.com/general/indexes/

Sensitivity analysis is also performed on the closed-form pricing formula, and is important for the risk management [15]. Following the discussion of the influence of the interest rate on yield to maturity, duration and convexity of risky mortgage loan, this study proposes that there is a positive relationship between the yield and the effects of volatility of the interest rate when the jump structure is included in state variable dynamics. Furthermore, there is a negative relationship between the mortgage duration (convexity) and the influence of the interest rate on the intensity of default rate and prepayment rate. A great degree of change in the state variable reduces the magnitude of risky mortgage duration and convexity. Finally, the speed of adjustment of interest rate enhances mortgage duration and convexity.

The rest of the paper is organized as follows: In Section II, we present the pricing model in both discrete and continuous time settings. In section III, we derive the closed-form pricing formula using the extended transform affine model. In Section IV, sensitivity analysis of mortgage loan yield, duration and convexity based on our pricing formula is provided. Finally, Section V presents conclusions.

II. Model

In this section, we present a model for pricing FRM, taking termination risks into consideration. The termination risks of a mortgage are prepayment risk and default risk. In this paper, we assume if the borrower prepay the mortgage, it means that he fully prepays this mortgage. We define M_0 as the initial mortgage principal, c the fixed coupon rate, and T the maturity of this mortgage. Because we assume the mortgage is a FRM, the payment is a constant for every payment date, and we assume the constant payment Y per unit time, which can be derived as follows:

$$Y = M_0(c/(1-\exp(-cT))). \tag{3}$$

To calculate the price of mortgage, we use a discrete time approximation to approximate the price of mortgage, and then derive the continuous formula of pricing mortgage. Assume the time interval of each payment date is Δt. The index i means the valuation point, where $i = 0, 1, ..., n$ and $n = T/\Delta t$. In the following, we are going to derive V_0, i.e., value of the mortgage. To do so, we consider time i as consisting of two phases. In the first phase, the borrower decides whether to default or to make the payment $Y\Delta t$. In the second phase, the borrower either pays the full amount of the outstanding principal M_i or to maintain the mortgage. We thus define V_i as the mortgage price at the end of the first phase of time i. Obviously, we have $V_n = 0$. It may also be seen that V_0 exactly defines the initial value of the mortgage. The value of M_i is $M_0(1-\exp[-c(T-i\Delta t)])/(1-\exp(-cT))$. The probability of prepayment at time i is P^P_{i+1}. When the borrower maintains this mortgage, there are two conditions. The borrower may default the mortgage with the probability P^D_{i+1} at time $i+1$, or may pay on time with the probability of $1-P^P_{i+1}-P^D_{i+1}$ at time $i+1$. If the borrower pays on time, the value of mortgage will be $Y\Delta t + V_{i+1}$ at time $i+1$. Thus, by discounting to time i, V_i will be $(Y\Delta t + V_{i+1})\exp(-r_{i+1}\Delta t)$. If the borrower defaults the mortgage, there will be some loss of the mortgage. We denote the loss rate of default as η_{i+1}, $0 < \eta_{i+1} \leq 1$. This rate is a random variable showing the partial loss of the mortgage value during the default. The value of the mortgage after default will be denoted as $(1-\eta_{i+1})(Y\Delta t + V_{i+1})$ at time $i+1$. Thus, by discounting to time i, V_i will be $(1-\eta_{i+1})(Y\Delta t + V_{i+1})\exp(-r_{i+1}\Delta t)$. As the mention above, we derive the general expression of V_i, where $i = 0, 1, ..., n-1$, as:

$$\begin{aligned} V_i =& E_i[(Y\Delta t + V_{i+1})(1-P^P_{i+1}-P^D_{i+1})\exp(-r_{i+1}\Delta t) + M_i P^P_{i+1} \\ &+ (Y\Delta t + V_{i+1})(1-\eta_{i+1})P^D_{i+1}\exp(-r_{i+1}\Delta t)] \end{aligned} \tag{4}$$

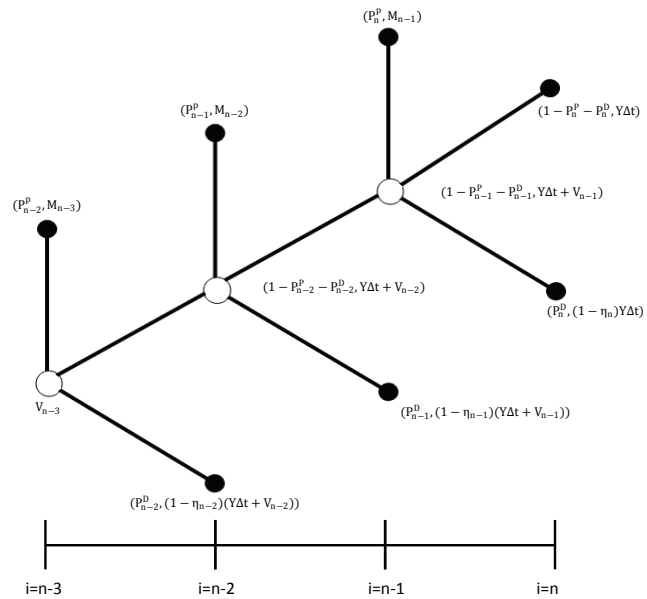

Fig. 3. This figure shows the possible behavior of borrower from $i = n-3$ to $i = n$. (,) express the values (probability, value of mortgage); the solid points represent this contract is exercised, and the hollow points represent borrower maintains this mortgage

Fig. 3 shows the possible decisions made by the borrower from $i = n-3$ to $i = n$. To use backward induction to derive initial mortgage value V_0, we take $i = n-1$ into equation 4 to express V_{n-1} as:

$$\begin{aligned} V_{n-1} =& E_{n-1}[Y\Delta t(1-P^P_n-P^D_n)\exp(-r_n\Delta t) \\ &+ Y\Delta t(1-\eta_n)P^D_n\exp(-r_n\Delta t) + M_{n-1}P^P_n]. \end{aligned} \tag{5}$$

The price means the value after payment at time $n-1$. The borrower may maintain, prepay, or default the mortgage at time $n-1$. When the borrower maintains the mortgage, the lender will receive the $Y\Delta t$ at time n. If the borrower prepays the mortgage, the lender will receive the outstanding principal M_{n-1}. We define the loss rate η of default as the ratio of the cash lender can receive after default over the price of the mortgage at the same time. Because when the borrower maintains the mortgage, the lender will receive the $Y\Delta t$ at time n, the cash flow of default event will be $(1-\eta_n)Y\Delta t$ at time n. When borrower prepays the mortgage, the lender receives the cash flow at time $n-1$. The lender receive receives the cash flow at time n when borrower maintains or defaults the mortgage. So, we need to discount the cash flow to time $n-1$ when borrower maintains or defaults the mortgage.

Let $Q_i = \exp(-r_i\Delta t)[(1-P_i^P-P_i^D)+(1-\eta_i)P_i^D]$. By approximating $\exp(x)$ by $(1+x)$ on small value of x using the Taylor series expansion [16], we can rewrite Q_i as:

$$\begin{aligned} Q_i &\cong 1-[r_i\Delta t+P_i^P-r_i\Delta t P_i^P+\eta_i P_i^D(1-r_i\Delta t)] \\ &\cong \exp[-(r_i\Delta t+P_i^P-r_i\Delta t P_i^P+\eta_i P_i^D(1-r_i\Delta t))]. \end{aligned} \quad (6)$$

Then we can rewrite equation (5) as:

$$V_{n-1} = E_{n-1}[Y\Delta t Q_n + M_{n-1}P_n^P]. \quad (7)$$

The value of the mortgage at time point $i = n-2$ is:

$$V_{n-2} = E_{n-2}[(Y\Delta t + V_{n-1})Q_{n-1} + M_{n-2}P_{n-1}^P]. \quad (8)$$

Substitute V_{n-1} from equation (7) into equation (8), and use property of expectation $E_t[E_{t+i}[.]] = E_t[.]$, we obtain a new equation:

$$\begin{aligned} V_{n-2} = E_{n-2}[&Y\Delta t Q_{n-1} + Y\Delta t Q_n Q_{n-1} \\ &+M_{n-1}P_n^P Q_{n-1} + M_{n-2}P_{n-1}^P]. \end{aligned} \quad (9)$$

Iterating to the initial point, we can obtain the initial mortgage value V_0 as:

$$\begin{aligned} V_0 =& E_0[Y\Delta t\sum_{i=1}^{n}(\prod_{j=1}^{i}Q_j)] \\ &+ E_0[\sum_{i=1}^{n}M_{i-1}P_i^P((\prod_{j=1}^{i}Q_j)/Q_i)], \end{aligned} \quad (10)$$

where $Q_0 = 1$, and$(\prod_{j=1}^{i}Q_j)/Q_i = Q_0(\prod_{j=1}^{i}Q_j)/Q_i = \prod_{j=1}^{i}Q_{j-1}$. Substitute Q_i by equation (6), we get

$$\begin{aligned} V_0 =& YE_0[\Delta t\sum_{i=1}^{n}\exp(-\sum_{j=1}^{i}(r_j\Delta t+P_j^P-r_j\Delta t P_j^P \\ &+\eta_j P_j^D(1-r_j\Delta t)))] + M_0/(1-\exp(-cT)) \\ &E_0[\sum_{i=1}^{n}(1-\exp(-c(T-(i-1)\Delta t)))P_i^P\exp(\\ &-(\sum_{j=1}^{i}(r_j\Delta t+P_j^P-r_j\Delta t P_j^P+\eta_j P_j^D(1-r_j\Delta t))) \\ &+((r_j\Delta t+P_j^P-r_j\Delta t P_j^P+\eta_j P_j^D(1-r_j\Delta t))))]. \end{aligned} \quad (11)$$

Following, we want to derive the continuous form of the initial mortgage value V_0. We model the default and prepayment probabilities as Poisson Processes with time-varying intensities. The conditional probabilities of default and prepayment in $[t, t+dt]$ is proportional to dt with the intensities of default and prepayment as coefficients. We define the intensities of default and prepayment as λ_t^D and λ_t^P, where $0 \le t \le T$. Then we can represent the conditional probabilities of default and prepayment as $P_t^D = \lambda_t^D dt$ and $P_t^P = \lambda_t^P dt$. The continuous form of the initial mortgage value V_0 can be expressed as follows:

$$\begin{aligned} \lim_{\Delta t\to 0} V_0 = Y\int_0^T & E_0[\exp(-(\int_0^t r_u du + \int_0^t \lambda_u^p du \\ &- \int_0^t \lambda_u^p r_u (du)^2 + \int_0^t \lambda_u^D \eta_u (1-r_u du)du))]dt + \\ & M_0/(1-\exp(-cT))\int_0^T E_0[(1-\exp(-c(T-t))) \\ & \lambda_t^p \exp -(\int_0^t r_u du + \int_0^t \lambda_u^p du - \int_0^t \lambda_u^p r_u (du)^2 + \\ & \int_0^t \lambda_u^D \eta_u (1-r_u du)du)\varsigma(t)]dt. \end{aligned} \quad (12)$$

Where $\varsigma(t) = \exp(r_t dt + \lambda_t^p dt - \lambda_t^p r_t (dt)^2 + \lambda_t^D \eta_t (1 - r_t dt)dt)$. Because $(r_t dt + \lambda_t^p dt - \lambda_t^p r_t (dt)^2 + \lambda_t^D \eta_t (1 - r_t dt)dt)$ is very small, we can use Taylor series to express $\varsigma(t)$ as:

$$\begin{aligned} \varsigma(t) &\cong 1 + (r_t dt + \lambda_t^p dt - \lambda_t^p r_t (dt)^2 + \lambda_t^D \eta_t (1-r_t dt)dt) \\ &\cong 1 + (r_t + \lambda_t^p + \lambda_t^D \eta_t)dt. \end{aligned}$$

Because $\varsigma(t)$ is in the integral of equation (12), we can apply 1 to approximate $\varsigma(t)$. According to the definition above, η_t is the random variable denoting the loss given default. Jokivuolle and Peura [14] shows that there is no significant difference on pricing mortgages both when the loss given default is a random variable or a constant. In that case, we can have $\eta_t = \eta$. Otherwise, $\lambda_u^p r_u (du)^2$ and $\lambda_u^D \eta_u r_u (du)^2$ are very small in real world, so we can neglect these terms. From the above, we can rewrite equation (12) as:

$$\begin{aligned} V_0 = Y\int_0^T & E_0[\exp(-\int_0^t (r_u + \lambda_u^p + \eta\lambda_u^D)du)]dt + \\ & M_0/(1-\exp(-cT))\int_0^T E_0[(1-\exp(-c(T-t))) \\ & \lambda_t^p \exp -(\int_0^t (r_u + \lambda_u^p + \eta\lambda_u^D)du)]dt. \end{aligned} \quad (13)$$

III. The closed-form pricing formula

The value of the mortgage equals the expectation of the future cash flow under the risk-neutral measure. To obtain the closed-form pricing formula, we set up a state variable describing the variation in the economy, which is fruitful in developing the tractable and conventional solution from Duffie et al. [11]. The affine jump-diffusion (AJD) is specified, which is a jump diffusion process for which the drift vector, covariance matrix and jump intensity all have affine dependence on the state variable related to the interest rate.

In this section, we introduce the interest rate process as the state variable in the economy. To characterize the financial events (triggers), Kou [13] proposed the interest rate process follows a double exponential jump diffusion process as follows:

$$dr_t = \kappa(\overline{r} - r_t)dt + \sigma dW_t + d(\sum_{i=1}^{N_t} \upsilon_i), \tag{14}$$

where κ denotes the speed of adjustment to revert to long-run mean, and is a positive constant. σ is the volatility of short term interest rate. $\overline{r}$ denotes the long-run mean of the interest rate. Kou [13] defined N_t as a Poisson process with intensity λ. υ_i denotes magnitude of a jump which is used to model unanticipated attack in macroeconomic environment. The behavior of υ is captured by an asymmetric double exponential distribution with the density

$$f_\upsilon(\nu) = p\eta_1 \exp(-\eta_1\nu)1_{\nu\geq 0} + q\eta_2 \exp(-\eta_2\nu)1_{\nu\leq 0}, \tag{15}$$

where p, $q \geq 0$, $p+q = 1$. W_t is a standard Brownian motion independent of N_t and $f_\upsilon(\nu)$. $1_{(.)}$ is an indicator function. Finally, σ is the volatility term, which is also constant.

Following Duffie et al. [11], this study further represents the intensity rates of prepayment and default, λ_t^P and λ_t^D, respectively, as given by:

$$d\lambda_t^P = -\lambda_1^P dr_t. \tag{16}$$

$$d\lambda_t^D = \lambda_1^D dr_t. \tag{17}$$

From the above equations, the intensity rates of prepayment and default are proportional to the interest rate, the state variable in this economy. Because an increased interest rate would result in decreased probability of prepayment, the intensity of prepayment is negatively related to interest rate. However, an increased interest rate would result in decreased price of house, which may cause an increased probability of default. So, we set the positive relation of intensity of default and interest rate. It is worth to note that the jump structure of the intensity rates resulting from the state variable fully captures the random arrival of non-financial prepayment and default. For simplicity, we ignore the constant drift term.

The closed-form formula is based on the previous results in the finance literature. By CIR dynamic process (Cox, Ingersoll, Ross) [17], we denote as, a random variable, whose expected value and variance as shown in (Heath et al.) [18] are given as:

$$\begin{aligned}\mu_t^r &= E_0[\theta_t] \\ &= f(0,t) + \sigma^2(t - 2(1-\exp(-\kappa t))/\kappa + \\ &\quad (1-\exp(-2\kappa))/2\kappa)/2\kappa^2 + E[(\sum_{i=1}^{N_t} \upsilon_i)],\end{aligned} \tag{18}$$

and

$$\begin{aligned}\Sigma_t^r &= \sigma^2(t - 2(1-\exp(-\kappa t))/\kappa + \\ &\quad (1-\exp(-2\kappa t))/2\kappa)/\kappa^2 + Var[(\sum_{i=1}^{N_t} \upsilon_i)].\end{aligned} \tag{19}$$

From above discussion, we begin to take the first line of expectation for the mortgage value by substituting equation (17) and equation (18) into equation (12) as follows:

$$\begin{aligned}&E_0[\exp(\int_0^t -(r_u + \lambda_u^P + \eta\lambda_u^D)du)] \\ &= \exp(-(\lambda_0^P + \eta\lambda_0^D))E_0[\exp(-(1-\lambda_1^P + \eta\lambda_1^D)\int_0^t r_u du)] \\ &= -\exp(\lambda_0^P + \eta\lambda_0^D - (1-\lambda_1^P + \eta\lambda_1^D)\mu_t^r + \\ &(1-\lambda_1^P + \eta\lambda_1^D)^2\Sigma_t^r/2 + \lambda t(\theta(-(1-\lambda_1^P + \eta\lambda_1^D)) - 1)),\end{aligned} \tag{20}$$

where $\theta(.)$ is the moment generation function of double exponential jump structure:

$$\theta(c) = p\eta_1/(\eta_1 - c) + q\eta_2/(\eta_2 - c). \tag{21}$$

Next, the extended transform (Duffie et al. [11]) is applied for the second line of equation (13) for the closed-form mortgage pricing formula. This is regarded as the following calculation:

$$\begin{aligned}&E_0[\lambda_t^P \exp(\int_t^T (r_u + \lambda_u^P + \eta\lambda_u^D)du)] \\ &= \exp(\alpha(t) + \beta(t)r_t)(A(t) + B(t)r_t),\end{aligned} \tag{22}$$

where $A(t)$ and $B(t)$ satisfy the following ordinary deferential equations (ODEs) [13]:

$$\begin{cases} -\dot{B}(t) = -\kappa B(t) \\ -\dot{A}(t) = \kappa\bar{r}B(t) + \beta(t)\sigma^2 B(t) + \lambda\nabla\theta(\beta(t))B(t), \end{cases} \tag{23}$$

with boundary conditions: $B(T) = -\lambda_1^P$, $A(T) = 0$. Before solving the above ODEs system, we need to find the solution for $\beta(t)$. Particularly, $\alpha(t)$ and $\beta(t)$ also need to satisfy the following ODEs [13]:

$$\begin{cases} \dot{\beta}(t) = (1-\lambda_1^P + \eta\lambda_1^D) + \kappa B(t) \\ \dot{\alpha}(t) = (\lambda_u^P + \eta\lambda_u^D) - \kappa\bar{r}\beta(t) + \beta(t)^2\sigma^2/2 - \lambda(\theta(\beta(t)) - 1), \end{cases} \tag{24}$$

with boundary conditions: $\alpha(T) = 0$, $\beta(T) = 0$. We lay out the formulations for pricing mortgage value:

$$\beta(t) = (1 - \lambda_1^P + \eta\lambda_1^D)(\exp(-\kappa(T-t)) - 1). \tag{25}$$

$$\begin{aligned}\alpha(t) = {} & (\lambda_0^P + \eta\lambda_0^D)t - \bar{r}(1 - \lambda_1^P + \eta\lambda_1^D)(\exp(-\kappa(T-t))/\kappa \\ & - t) - (\alpha^2(1 - \lambda_1^P + \eta\lambda_1^D)/(2\kappa^2))(\exp(-2\kappa(T-t))/ \\ & (2\kappa) - 2\exp(-\kappa(T-t))/\kappa + t) - M(t) + c_1,\end{aligned} \tag{26}$$

where

$$\begin{aligned}c_1 = {} & M(T) - (\lambda_0^P + \eta\lambda_0^D)T + \bar{r}(1 - \lambda_1^P + \eta\lambda_1^D)(1/\kappa - T) \\ & + (\alpha^2(1 - \lambda_1^P + \eta\lambda_1^D)/(2\kappa^2))(-3/(2\kappa) + T),\end{aligned} \tag{27}$$

and

$$M(t) = \int_0^t (p\eta_1/(\eta_1 - \beta(u)) + q\eta_2/(\eta_2 - \beta(u)))du. \tag{28}$$

Take $\beta(t)$ into equation (23), we further obtain the following results:

$$B(t) = -\lambda_1^P \exp(-\kappa(T-t)). \tag{29}$$

$$\begin{aligned}A(t) = {} & \bar{r}\lambda_1^P \exp(-\kappa(T-t)) + \lambda_1^P\sigma^2(1 - \lambda_1^P + \eta\lambda_1^D)^2/(\kappa)^2 \\ & (\exp(-3\kappa(T-t))/(3\kappa) - \exp(-2\kappa(T-t))/\kappa \\ & - exp(-\kappa(T-t))/\kappa) - N(t) + c_2.\end{aligned} \tag{30}$$

$$c_2 = -\bar{r}\lambda_1^P + \lambda_1^P\sigma^2(1 - \lambda_1^P + \eta\lambda_1^D)^2/(3\kappa^3) + N(T). \tag{31}$$

$$\begin{aligned}N(t) = {} & \int_0^t (p\eta_1/(\eta_1 - \beta(u))^2 + q\eta_2/(\eta_2 - \beta(u))^2) \\ & \exp(-\kappa(T-u))du.\end{aligned} \tag{32}$$

Thus, we have the closed-form pricing formula for mortgage loan:

$$\begin{aligned}V_0 = {} & Y\int_0^T (\exp(-(\lambda_0^P + \eta\lambda_0^D)t - (1 - \lambda_1^P + \eta\lambda_1^D)\mu_t^r + \\ & (1 - \lambda_1^P + \eta\lambda_1^D)\Sigma_t^r + \lambda_t(\theta(-(1 - \lambda_1^P + \eta\lambda_1^D)) - 1)))dt \\ & + M_0/(1 - \exp(-cT))\int_0^T ((1 - \exp(-(T-t))) \\ & \exp(\alpha(t) + \beta(t)r_0 - (\lambda_0^P + \eta\lambda_0^D)t)(A(t) + B(t)r_0))dt.\end{aligned} \tag{33}$$

IV. Risk Management

A. The yield to maturity, duration and convexity for mortgage loan with prepayment and default risk

The risk-adjusted yield to maturity (YTM), R, of the mortgage loan is defined as the discount rate that equates the present value of a security's future cash flow to its initial value, as given by:

$$V_0 = Y\int_0^T \exp(-Rt)dt. \tag{34}$$

Therefore, we obtain the relationship between YTM and mortgage loan fair value such that

$$\begin{aligned}& Y\exp(-Rt)dt = \\ & \quad Y\exp(-(\lambda_0^P + \eta\lambda_0^D)t - (1 - \lambda_1^P + \eta\lambda_1^D)\mu_t^r \\ & \quad + (1 - \lambda_1^P + \eta\lambda_1^D)\Sigma_t^r + \lambda_t(\theta(-(1 - \lambda_1^P + \eta\lambda_1^D)) \\ & \quad - 1)) + M_0/(1 - \exp(-cT))(1 - \exp(-(T-t))) \\ & \quad \exp(\alpha(t) + \beta(t)r_0 - (\lambda_0^P + \eta\lambda_0^D)t)(A(t) + B(t)r_0).\end{aligned} \tag{35}$$

The yield of a risky mortgage is

$$\begin{aligned}R = {} & -\log(\exp(-(\lambda_0^P + \eta\lambda_0^D)t - (1 - \lambda_1^P + \eta\lambda_1^D)\mu_t^r \\ & + (1 - \lambda_1^P + \eta\lambda_1^D)\Sigma_t^r + \lambda_t(\theta(-(1 - \lambda_1^P + \eta\lambda_1^D)) \\ & - 1)) + M_0/(Y(1 - \exp(-cT)))(1 - \exp(-(T-t))) \\ & \exp(\alpha(t) + \beta(t)r_0 - (\lambda_0^P + \eta\lambda_0^D)t)(A(t) + B(t)r_0))/t.\end{aligned} \tag{36}$$

The above formula describes the effect of the significant parameters of our model on the YTM. Particularly, the sensitivity analysis could be performed on this foundation, which also convey some information to participator in the risk management field.

The duration of the mortgage loan with prepayment and default risks is explored in the following way. First, the risk-adjusted duration is defined as

$$D = -(1/V_0)(\partial V_0/\partial R), \tag{37}$$

where R denotes the YTM. The duration can be also derived as follows:

$$D = -(1/V_0)Y\int_0^T t\exp(-Rt)dt = \int_0^T t\omega_t dt, \tag{38}$$

where $\omega_t = \exp(-Rt)/V_0 = \exp(-Rt)/\int_0^T \exp(-Rt)dt$, which represents the weight of cash flows at time t. Next, the definition of the convexity for a risky mortgage is

$$C = (1/V_0)(\partial^2 V_0/\partial R^2), \tag{39}$$

where $C = (1/V_0)Y\int_0^T t^2\exp(-Rt)dt = \int_0^T t^2\omega_t dt$.

B. Sensitivity Analysis

This subsection numerically illustrates the influence of the parameters on the mortgage yield, duration, and convexity for the closed-pricing formula. For sensitivity analysis, we have to set up some parameters. We assume the mortgage is a 20 years contract with $M_0 = 1$ million. Let $\lambda_0^P = 0.005$, $\lambda_0^D = 0.005$, $\lambda_1^P = 0.05$, $\lambda_1^D = 0.05$, $\kappa = 0.5$, $\eta = 0.7$, $\sigma = 0.1$, $\bar{r} = 0.02$, $r_0 = 0.02$, $\eta_1 = 0.01$, $\eta_2 = 0.01$, $\lambda = 0.001$, $c = 0.05$, $p = 0.4$, and $q = 0.6$. Under these settings, we can obtain $V_0 = 1,161,162$, $R = 0.0327$, $D = 8.919$, and $C = 112.179$. The initial value of such mortgage without the consideration of termination risks is $1,304,561$. We can observer that under the consideration of termination risk, the mortgage value decreases $143,399$. Furthermore, the Fig. 4 shows how the mortgage value changes with coupon rate c. We can see that the mortgage value increase with the bigger coupon rate. Moreover, when the coupon rate equal the yield to maturity, the mortgage value will equal the initial outstanding. So, when the banks or investment managers want to lend or invest such mortgage, they should ask the coupon rate higher than YTM to make sure they will not get loss on the issue date.

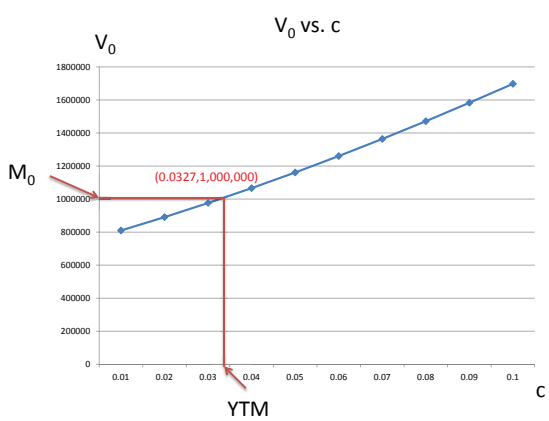

Fig. 4. Initial mortgage value for risky mortgage loan with respect to different coupon rate.

Then, we want to analyse how these three values change with four parameters, λ_1^P, λ_1^D, κ, and σ. The key task for portfolio manager is to explore the implication of the duration and convexity of their mortgage holding. Additionally, the main determinant of the investment decision is the yield, which is also considered in the following way.

Fig. 5 and Fig. 6 illustrate the influences of the interest rate (state variable) on the yield for the holding of risky mortgage loan. These figures show that a higher yield will be required when the effect of interest rate on the default (the value of λ_1^D) increases, and when the effect of interest rate on prepayment intensities (value of λ_1^P) decreases.

The Fig. 7 illustrates the yield as a function of the level of volatility of interest rate. A larger volatility implies that

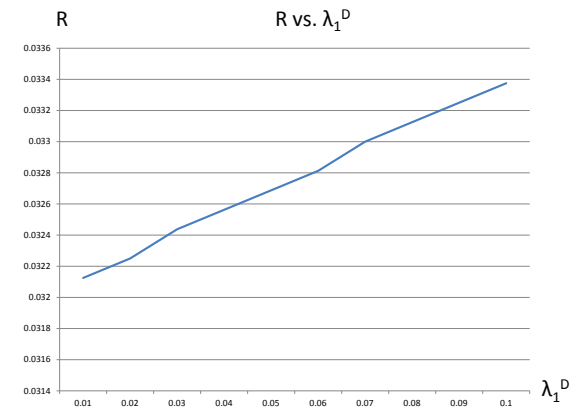

Fig. 5. Yield to maturity for risky mortgage loan with respect to different levels of the influence of interest rate on the intensity of default rate.

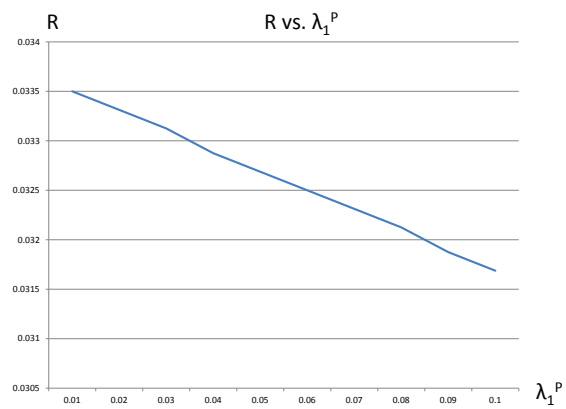

Fig. 6. Yield to maturity for risky mortgage loan with respect to different levels of the influence of interest rate on the intensity of prepayment rate.

high required rate is requested when there is a great degree of change in the state variable, reducing risky mortgage loan value. Additionally, the Fig. 8 shows that an effect related to how adjustment speed to revert to long-term mean influence the yield of risky mortgage loan. Larger speed to revert to long-term mean of the state variable enhances the required rate for holding the mortgage loan. Remarkably, the yield converges to a constant as $\kappa \to \infty$.

To provide the implication of the duration for risk management, the results are presented in Fig. 9 and Fig 10. These figures present views related to how the influences of the interest rate (state variable) on the yield influences the duration of mortgage loan. Increasing λ_1^D decreases the mortgage duration, and increasing λ_1^P increases the mortgage duration. It is worthy to note that our finding is similar to Chance [19] and Derosa et al. [20], the influences of interest

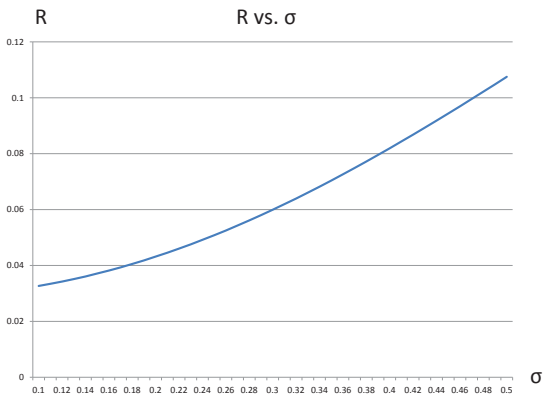

Fig. 7. Yield to maturity for risky mortgage loan with respect to different levels of the volatility of interest rate (σ).

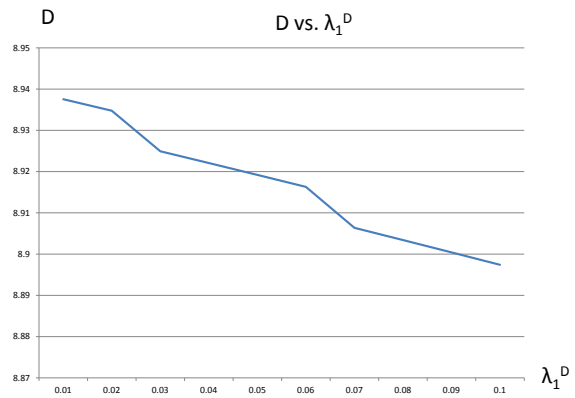

Fig. 9. Duration for risky mortgage loan with respect to different levels of the influence of interest rate on the intensity of default rate.

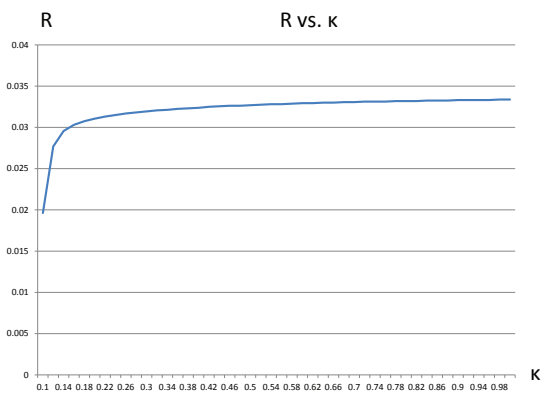

Fig. 8. Yield to maturity for risky mortgage loan with respect to different levels of the adjust speed of revert to long-term mean (κ).

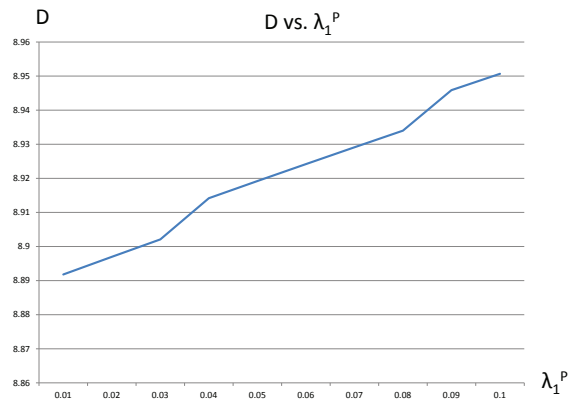

Fig. 10. Duration for risky mortgage loan with respect to different levels of the influence of interest rate on the intensity of prepayment rate.

rate on default will lead to smaller mortgage duration.

The Fig. 11 shows the duration as a function of the level of volatility of interest rate. The mortgage duration is inversely proportional to the volatility of the interest rate. This result is consistent with Tsai et al. [10]. The Fig. 12 shows that there is also a negative relationship between the duration and the speed to revert to long-term mean. Similarly, the mortgage duration converges to a constant as $\kappa \to \infty$.

Furthermore, we find that the decreasing curve of the mortgage convexity with respect to the positive influence of the interest rate on the intensity of default rate, and negative influence of the interest rate on the intensity of prepayment rate, based on our closed-form pricing formula, from Fig. 13 and Fig. 14. Noticeably, the result in Fig. 15 is similar to Fig. 11, and the result in Fig. 16 is similar to Fig. 12.

V. Concluding Remarks and Future Work

This study provides an effective pricing model for the valuation of FRM which takes into account uncertainties in borrowers' default and prepayment behaviors. We derive a closed-formed formula to compute the mortgage price and the risk measures. The major difference between our pricing model and previous studies is that, in our model, mortgage loan dynamics resulting from prepayment is time-varying with the changing in interest rate. We use a backward recursion method to value mortgage loan by considering outstanding principal as the prepayment value. In order to account for the financial events, our formula includes jump structure in the state variable process, also providing the solution to the problem of mortgage hedging. The duration and the convexity of fixed-income securities are very important hedging tools for risk management. Particularly, the

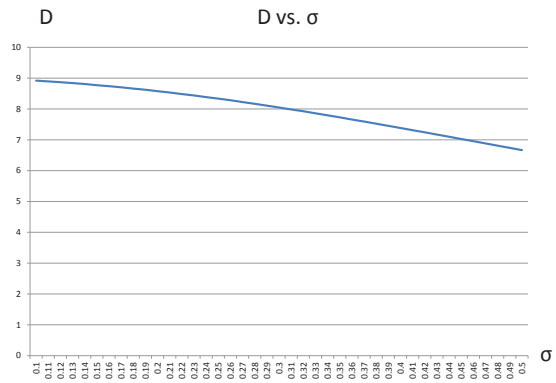

Fig. 11. Duration for risky mortgage loan with respect to different levels of the volatility of interest rate (σ).

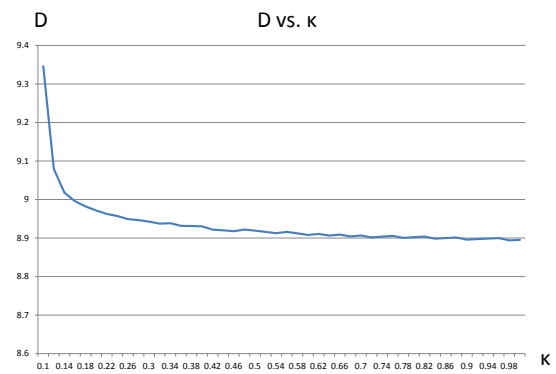

Fig. 12. Duration for risky mortgage loan with respect to different levels of the adjustment speed of revert to long-term mean (κ).

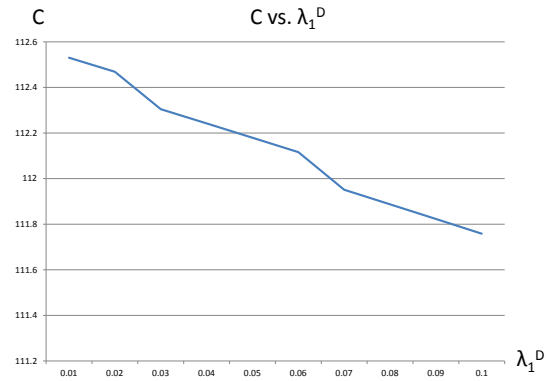

Fig. 13. Convexity for risky mortgage loan with respect to different levels of the influence of interest rate on the intensity of default rate.

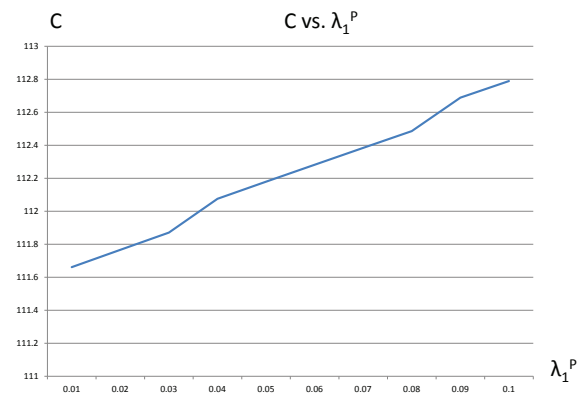

Fig. 14. Convexity for risky mortgage loan with respect to different levels of the influence of interest rate on the intensity of prepayment rate.

duration of the interest-rate-sensitive securities determines the hedging position for offsetting the loss due to the uncertainty of interest rate. In other words, we propose the practical implementation of pricing risky mortgage loan to relax the assumption that the prepayment behavior of debtor is constant. Finally, on the economic side, we provide the implication in the relationship between the risk measures (e.g. duration and convexity) and the changing of interest rate for banks and investment managers.

In this paper, we assume the intensity rates of prepayment and default are proportional to only one state variable in economy. However, the probabilities of prepayment and default could be expressed by other macro economic indices, such as house price index. We can include more state variables in our model in our further research. Otherwise, using real world data to test our closed-formed formula is another important future work.

VI. Discussion

In this study, we focus only on model generation of mortgage pricing. We believe that unknown parameters of the model may be estimated using statistical estimation methods, e.g., the maximum-likelihood estimation technique. However, it is beyond the scope of the current research to study the problem of estimating the unknown parameters. We would like to study further in the future the problem of validating the mortgage pricing model and to estimate the model parameters based on real data. As for borrower's prepayment behavior, we assume when prepayment occurs, the borrower fully prepay the mortgage. How to deal with the partial prepayment is also an interesting future work.

The result of Fig. 11 shows the negative relation between duration and volatility of the interest rate. This result is

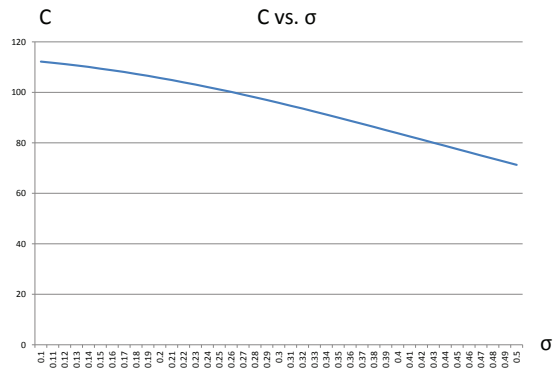

Fig. 15. Convexity for risky mortgage loan with respect to different levels of the volatility of interest rate (σ).

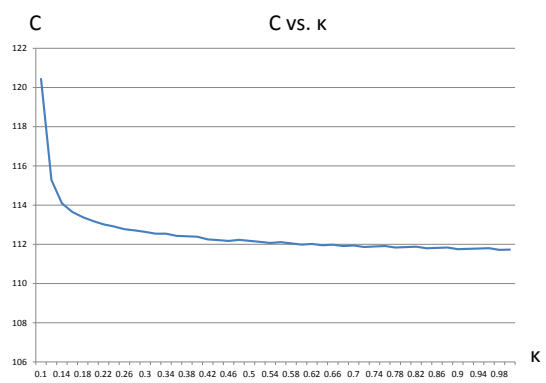

Fig. 16. Convexity for risky mortgage loan with respect to different levels of the adjustment speed of revert to long-term mean (κ).

in accordance with those presented by Tsai et al (2009). However, we do not have an economic intuition to explain the result. It is also left for interested readers for further studies.

In reality, a borrower tends to choose default if house price is lower than the outstanding principal. Thus, the default probability will decrease over time because the outstanding principal becomes smaller. However, the definition of default intensity in equation (17) does not reflect this reality. This is also an interesting problem to study further.

To simplify the derivation, we use first order Taylor series expansion to calculate Q_i and $\varsigma(t)$. We use the parameters of section IV to consider if the simplifications is reasonable. Let Δt to be one month, $r_i = 0.02$, $P_i^P = P_i^D = \lambda_0^P \Delta t$, and $\eta_i = 0.7$. First, we have to check if the following three values are small enough to use first order Taylor series expansion.

1.$r_i \Delta t = 0.0017$
2.$(r_i \Delta t + P_i^P - r_i \Delta t P_i^P + \eta_i P_i^D (1 - r_i \Delta t) = 0.0024$
3.$r_t dt + \lambda_t^p dt - \lambda_t^p r_t (dt)^2 + \lambda_t^D \eta_t (1 - r_t dt) dt = 0.0024$

These three values are all much smaller than one. Thus, the approximations of first order Taylor series expansion are reasonable. Let's also consider the approximation error in computing Q_i. Note that we simplify $Q_i = \exp(-r_i \Delta t)[(1 - P_i^P - P_i^D) + (1 - \eta_i) P_i^D]$ as $\hat{Q}_i = \exp[-(r_i \Delta t + P_i^P - r_i \Delta t P_i^P + \eta_i P_i^D (1 - r_i \Delta t))$. Thus, we define the approximation error as

$$Error = abs((Q_i - \hat{Q}_i)/Q_i).$$

The error of Q_i in this example is $1.43 * 10^{-6}$. We also simplify $\varsigma(t) = \exp(r_t dt + \lambda_t^p dt - \lambda_t^p r_t (dt)^2 + \lambda_t^D \eta_t (1 - r_t dt) dt)$ as $\hat{\varsigma}(t) = 1 + (r_t + \lambda_t^p + \lambda_t^D \eta_t) dt$. Using the same parameters, the error is $1.63 * 10^{-6}$. The errors due to Taylor series expansion are small enough to be ignored in the derivation.

Acknowledgements

We want to thank the Editor Chen, Meng Chang and two anonymous reviewers of Journal of Information Science for glad comments and suggestions on earlier drafts.

This research was supported in part by the Foresight Taiwan Program and National Science Council of Taiwan under grants: NSC 102-2221-E-001-015-MY3, NSC 101-2221-E-001-003, and 102-2221-E-001 -015 -MY3.

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