A novel method for ranking generalized fuzzy numbers*

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In most of the commonly used approaches, the ranking of generalized fuzzy numbers is calculated based on the only one criterion i.e., rank criteria. However, there are some weaknesses associated with this criterion. In this paper, we discuss the insufficiency of considering only rank for the ordering of generalized fuzzy numbers. Some shortcomings in existing ranking methods have been pointed out. To overcome these drawbacks, we present a new ranking method which considers not only rank, but also some other criteria, such as mode, divergence and left or right spread of generalized fuzzy numbers, which are very important for the ordering of generalized fuzzy numbers. Several numerical examples are provided to illustrate the superiority of the proposed method. Some results in the support of the proposed method are also presented.

Keywords: fuzzy sets, generalized fuzzy numbers, ranking function, ordering relation

1. INTRODUCTION

In a classical contest real number $R$ can be linearly ordered by $\geq$ however, the ranking of fuzzy numbers cannot be executed in such a way. Since fuzzy numbers are represented by possibility distributions, they can overlap with each other and it is difficult to determine clearly whether one fuzzy number is larger or smaller than the other. From the application viewpoint, the ranking of fuzzy numbers plays a main role in real life problems involving decision-making, clustering, optimization, transportation problems, etc. [2, 9, 10, 23] In order to rank fuzzy numbers, one fuzzy number needs to be evaluated and compared to others but this may not be so easy. The literature over the past few decades has proposed numerous methods for ranking fuzzy numbers. Some of them are legendary for ranking fuzzy numbers, such as maximizing sets and minimizing sets, centroid points, and distance minimization. Jain [11] proposed the first ranking method using the maximizing set to order fuzzy numbers for selecting an optimal alternative. Yager [23] presented the centroid-index method, Dubios and Prade [9] used the maximizing set to order fuzzy numbers, and Chen [5] proposed the maximizing and minimizing set approach. Chu and Tsao [8] ranked fuzzy numbers with an area between the centroid point and original point. Wang et al. [21] presented the centroid of fuzzy numbers. Abbasbandy and Asady [1] suggested sign distance, while Asady and Zendehnam [4] proposed the distance minimization method.

Many researchers have recently employed maximizing set and minimizing set and the concept of a centroid point as the basis for comparing and ranking fuzzy numbers [3, 10, 17, 19, 20]. These methods have mainly concern the correlation between L-R areas and centroid point of a fuzzy number. Wang et al. [19] defined the L-R deviation degree of a fuzzy number and came up with the ranking rule, in which the larger the left deviation...
degree and the smaller the right deviation degree are, the larger the fuzzy number is. Asady [3] and Nejad Mashinchi [17] redefined the L-R deviation degree of a fuzzy number to overcome the shortcomings of Wang et al. [19]. However, most deviation degree approaches still display the same limitations due to the neglected decision maker’s attitude, the incoherent transfer coefficient formula, and the unreliable ranking index computation. More ranking literature can be found in [6, 13, 14, 15, 16, 17, 18, 21].

However, in all the existing ranking methods only rank criteria are used to compare the fuzzy numbers and generalized fuzzy numbers. But, we observe that the rank is not sufficient criteria to find the correct ordering of the generalized fuzzy numbers, so generalized fuzzy numbers needs further evaluation for the reasonable ordering. In this paper, we pointed out the shortcomings in existing ranking methods. In order to remove these shortcomings, we present a new ranking method which considers not only rank criterion, but also mode, divergence and left or right spread criteria of generalized fuzzy numbers. Some propositions which support the proposed method are also presented.

The rest of the paper is organized as follows: Section 2, review some basic definitions and the existing ranking method. In Section 3, the shortcomings of Chen and Sanguansats method [6] and improved Xu et al. method [22] for the ranking of generalized fuzzy numbers are discussed. With the help of counter examples it is pointed out that Chen and Sanguansats method [6], and Xu et al. [22] do not provide the correct ordering for all the cases. In section 4, some new results related to rank, mode, divergence left and right spread criteria of generalized fuzzy numbers. Based on these results a new method is developed for the ranking of generalized fuzzy numbers. The results of the proposed method with some existing ranking methods are compared in Section 5. Section 6 draws conclusions.

2. PRELIMINARIES

This section reviews some basic definitions of generalized fuzzy numbers and some existing ranking methods.

2.1 Basic Definitions of Generalized Trapezoidal Fuzzy Numbers

**Definition 1.** [9] A fuzzy number \( A = (a, b, c, d; w_A) \) is described as any fuzzy subset of the real line \( R \) with membership function \( f_A(x) \) that possesses the following features:

1. \( f_A(x) : R \rightarrow [0, w_A] \) is continuous,
2. \( f_A(x) = 0 \), for all \( x \in (-\infty, a] \),
3. \( f_A(x) \) is strictly increasing on \( x \in [a, b) \),
4. \( f_A(x) = 1 \), for \( x = [b, c] \)
5. \( f_A(x) \) is strictly decreasing on \( x \in (b, d] \),
6. \( f_A(x) = 0 \), for all \( x \in (-\infty, d] \).

**Definition 2.** [9] A trapezoidal fuzzy number \( A = (a, b, c, d; w_A) \) is called generalized trapezoidal fuzzy number, if it is described as any fuzzy subset of the real line \( R \) with membership function \( f_A(x) \) which is expressed as:
When $b = c$, the generalized trapezoidal fuzzy number is reduced to a generalized triangular fuzzy number and can be denoted by $A = (a, b, d; w_A)$

### 2.2 Review of Existing Ranking Approaches

#### 2.2.1 Chen and Sanguansats method [6]

Let $A = (a_1, b_1, c_1, d_1; w_A)$ and $B = (a_2, b_2, c_2, d_2; w_B)$ be two generalized fuzzy numbers, then use the following steps to compare $A$ and $B$.

**Step 1:** Standardize each generalized fuzzy number $A$ and $B$ into $A^*$ and $B^*$ as follows:

$$A = \left( \frac{a_i}{k}, \frac{b_i}{k}, \frac{c_i}{k}, \frac{d_i}{k}; w_A \right) = (a_i^*, b_i^*, c_i^*, d_i^*; w_A)$$

Where, $k = \max(\lVert a_i \rVert, \lVert b_i \rVert, \lVert c_i \rVert, \lVert d_i \rVert)$, $\lVert a_i \rVert, \lVert b_i \rVert, \lVert c_i \rVert, \lVert d_i \rVert$ denoted the absolute value of $a_i, b_i, c_i, d_i$ respectively, and $\lVert a_i \rVert, \lVert b_i \rVert, \lVert c_i \rVert, \lVert d_i \rVert$ denoted the upper bound of $|a_i|, |b_i|, |c_i|, |d_i|$ respectively.

$$B = \left( \frac{a_2}{k}, \frac{b_2}{k}, \frac{c_2}{k}, \frac{d_2}{k}; w_B \right) = (a_2^*, b_2^*, c_2^*, d_2^*; w_B)$$

Where, $k = \max(\lVert a_i \rVert, \lVert b_i \rVert, \lVert c_i \rVert, \lVert d_i \rVert)$, $\lVert a_i \rVert, \lVert b_i \rVert, \lVert c_i \rVert, \lVert d_i \rVert$ denoted the absolute value of $a_2, b_2, c_2, d_2$ respectively, and $\lVert a_i \rVert, \lVert b_i \rVert, \lVert c_i \rVert, \lVert d_i \rVert$ denoted the upper bound of $|a_2|, |b_2|, |c_2|, |d_2|$ respectively.

**Step 2:** Calculate the areas $\text{Area}_{A^*}$ and $\text{Area}_{B^*}$, respectively, which denote the areas from the membership function curves of $\mu_A^*$ and $\mu_B^*$ respectively, to the membership function curve of the generalized fuzzy number $(-1, -1, -1; w_A)$ respectively, where

$$\mu_A^* = w_A \frac{(x - a_i^*)}{(b_i^* - a_i^*)}, a_i^* \leq x < b_i^*$$

$$\mu_B^* = w_A \frac{(x - d_i^*)}{(c_i^* - d_i^*)}, c_i^* < x \leq d_i^*$$

and
\[
\text{Area}_{A_L}^+ = \frac{w_A}{2} \left( \frac{a_1' + 1 + b_1'}{2} \right) \\
\text{Area}_{A_R}^+ = \frac{w_A}{2} \left( \frac{a_1' + 1 + d_1'}{2} \right)
\]  
\hspace{1cm} (5)

Then, calculate the areas \( \text{Area}_{A_L}^- \) and \( \text{Area}_{A_R}^- \), respectively, which denote the areas from the membership function curves of \( \mu_A^L \) and \( \mu_A^R \) defined in Eqs. (1) and (2), respectively, to the membership function curve of the generalized fuzzy number \((-1, -1, -1, -1; w_A)\), where

\[
\text{Area}_{A_L}^- = \frac{w_A}{2} \left( \frac{1 - a_1'}{(1 - b_1')} \right) \\
\text{Area}_{A_R}^- = \frac{w_A}{2} \left( \frac{1 - c_1'}{(1 - d_1')} \right)
\]  
\hspace{1cm} (7)

Similarly, for generalized fuzzy set \( B \)

\[
\text{Area}_{B_L}^+ = \frac{w_A}{2} \left( \frac{a_2' + 1 + b_2'}{2} \right) \\
\text{Area}_{B_R}^+ = \frac{w_A}{2} \left( \frac{a_2' + 1 + d_2'}{2} \right)
\]  
\hspace{1cm} (9)

\[
\text{Area}_{B_L}^- = \frac{w_A}{2} \left( \frac{1 - a_2'}{(1 - b_2')} \right) \\
\text{Area}_{B_R}^- = \frac{w_A}{2} \left( \frac{1 - c_2'}{(1 - d_2')} \right)
\]  
\hspace{1cm} (11)

**Step 3:** Calculate the values

\[
XI_{A'} = \text{Area}_{A_L}^+ + \text{Area}_{A_R}^+ \\
XD_{A'} = \text{Area}_{A_L}^- + \text{Area}_{A_R}^-
\]  
\hspace{1cm} (13)

and

\[
XI_{B'} = \text{Area}_{B_L}^- + \text{Area}_{B_R}^+ \\
XD_{B'} = \text{Area}_{B_L}^+ + \text{Area}_{B_R}^-
\]  
\hspace{1cm} (15)

**Step 4:** Calculate the ranking score \( \text{Score}(A') \) and \( \text{Score}(B') \) of each generalized trapezoidal fuzzy number \( A' \) and \( B' \) as follows:

\[
\text{Score}(A') = \frac{XI_{A'} - XD_{A'}}{XI_{A'} + XD_{A'} + (1 - w_A)}
\]  
\hspace{1cm} (17)

and

\[
\text{Score}(B') = \frac{XI_{B'} - XD_{B'}}{XI_{B'} + XD_{B'} + (1 - w_B)}
\]  
\hspace{1cm} (18)

where, \( \text{Score}(\bullet) \). The larger is the value of \( \text{Score}(A') \), the better the ranking of \( A' \). Based on Eqs (17), (18), it can be seen that \( XI_{\bullet}, \) and \( XD_{\bullet} \), are used for weighting the maximal and minimal possible values of the universe of discourse of generalized fuzzy num-
bers 1 and \(-1\), respectively. This mean that larger is the value of \(XI_{\alpha}\), the closer the value of \(Score(\bullet)\) is closer to 1; larger is the values \(XD_{\alpha}\), the closer the value of \(Score(\bullet)\) is closer to \(-1\). Moreover, Chen and Sanguansat's method consider the factor on \(y\)-axis as less influential to the ranking score of generalized fuzzy numbers, \((1-w_\alpha)\) is added into the denominator in Eq. (17), i.e., \(|Score(A^\alpha)|\) decreases with the decrease in the value of \(w_\alpha\).

2.2.2 Xu et al. improved method [22]

The modified method of Xu et al. is describe as follows:

**Step 1:** Transform each generalized fuzzy number \(A_i(1 \leq i \leq n)\) into standardized generalized fuzzy number \(A^*_i\), as defined in Eq. (1).

**Step 2:** Calculate the area of each standardized generalized fuzzy number \(A^*_i\), as defined in Eqs. (5)-(8).

**Step 3:** Calculate the values \(XI_{A^*_i}\) and \(XD_{A^*_i}\) each standardized generalized fuzzy number \(A^*_i\), as defined in Eqs. (13) and (14).

**Step 4:** Calculate the ranking score \(Score(A^*_i)\) of each standardized generalized fuzzy number \(A^*_i\), as defined in Eq. (17), the larger is the value of \(Score(A^*_i)\) the better the ranking of \(A^*_i\).

**Step 5:** For generalized fuzzy number set: \(R = \{A^*_i | Score(A^*_i) = 0.1 \leq i \leq n\}\), If \(|R| = 1\) where, \(|R|\) denotes the cardinality of \(R\), then rank the generalized fuzzy number in \(R\) according to their heights \(w_{A^*_i}\). The larger the value of \(w_{A^*_i}\), the better the ranking of \(A^*_i\).

3. DRAWBACKS IN EXITING RANKING METHODS

3.1 Shortcomings in Chen and Sanguansat's Method [6]

With the help of some examples we can see that there are some drawbacks in Chen and Sanguansat's method (2011).

Suppose \(A^\alpha = (a_1, b_1, c_1, d_1; w_\alpha)\), where \(-1 \leq a_1 \leq b_1 \leq c_1 \leq d_1 \leq 1\) and \(a_1 + b_1 + c_1 + d_1 = 1\) or \(w_\alpha = 0\) then \(Score(A^\alpha) = 0\). It means that, if center of gravity of \(A^\alpha\) on the \(x\)-axis is 0, then \(Score(A^\alpha) = 0\). Therefore, the ranking by Chen and Sanguansat's method gives unreasonable results. To support our argument following examples are given. Some more examples can be seen in Section 5.

**Example 1.** \(A = (-0.5, -0.3, -0.3, -0.1; 1)\), \(B = (0, 0, 0, 0; 1)\) and \(C = (0, 0, 0, 0.8)\) be generalized fuzzy numbers, then according to Chen and Sanguansat's Method [6] \(A, B\) and \(C\) are equal. But if we look at the structure of \(A\) and \(B\) they are not seems to be equal.
Example 2. \( A = (-0.2, 0.0, 0.2; 0.8) \) and \( B = (-0.2, 0.0, 0.2; 1) \) be generalized fuzzy numbers, then according to Chen and Sanguansat’s Method [6] \( A \) and \( B \) are equal. But if we look at the structure of \( A \) and \( B \) they are not seems to be equal.

3.2 Shortcomings in Xu et al. [22]

Intuitively, we observe that the ordering of generalized fuzzy numbers does not depend only on the rank of fuzzy numbers. Rank is not a sufficient criterion to compare generalized fuzzy numbers, there are some other criterion which are very important and must be consider for the ordering of generalized fuzzy numbers.

We observe that, if \( a_i + b_i + c_i + d_i = 1 \) such that the set of all generalized fuzzy numbers: \( R = \{ A_i | Score(A_i) = 0.1 \leq i \leq n \} \), where \( |R| > 1 \) denotes the cardinality of \( R \), then rank the generalized fuzzy number in \( R \) according to their heights \( w_{A_i} \). The larger the value of \( w_{A_i} \), the better the ranking of \( A_i \). According to Xu et al. method, the generalized fuzzy numbers for which \( a_i + b_i + c_i + d_i = 0 \) with similar heights \( w_{A_i} \), have same order. But, when we look the structure of these type of generalized fuzzy numbers we found the numbers are not equal. It means only rank is not a sufficient criterion to compare generalized fuzzy numbers, therefore, some other criterion must be consider to find the reasonable results.

Following are some examples for which Xu et al. [22] modified ranking method unable to give the reasonable results. More examples can be seen in Section 5.

Example 3. Let \( A = (-0.2, -0.1, 0.1, 0.2; 0.4) \) and \( B = (-0.1, 0.0, 0.1; 0.4) \) be two generalized fuzzy numbers, then according to Xu et al. [21] \( A \) and \( B \) are equal. But if we look at the structure of \( A \) and \( B \) they are not seems to be equal, as shown in Fig. 1.

Example 4. Let \( A = (-0.5, -0.2, 0.3, 0.4; 0.6) \) and \( B = (-0.6, -0.3, 0.4, 0.5; 0.6) \) be two generalized fuzzy numbers, then according to Xu et al. [22], \( A \) and \( B \) are equal. But if we look at the structure of \( A \) and \( B \) they are not seems to be equal, as shown in Fig. 2.
4. PROPOSED METHOD FOR RANKING OF GENERALIZED FUZZY NUMBERS

In this section, first we present some basic results and then we propose a new method for the ordering of generalized fuzzy numbers which consider five criteria, such as rank, mode, divergence, left or right spread and height of generalized fuzzy sets.

Definition 3. Let \( A = (a, b, c, d; w) \) be any generalized fuzzy number, then we define;

1. \( \mathfrak{R}(A) = \frac{w(a + b + c + d)}{4} \) [13],
2. \( \text{mode } (A) = \frac{w(b + c)}{2} \),
3. \( \text{divergence } (A) = w(d - a) \),
4. \( \text{Left spread } (A) = w(b - a) \),
5. \( \text{Right spread } (A) = w(d - c) \).

Proposition 1. Let \( A = (a_1, b_1, c_1, d_1; w_1) \) and \( B = (a_2, b_2, c_2, d_2; w_2) \) be two generalized fuzzy numbers such that

1. \( \mathfrak{R}(A) = \mathfrak{R}(B) \),
2. \( \text{mode } (A) = \text{mode } (B) \),
3. \( \text{divergence } (A) = \text{divergence } (B) \)

then

a. Left spread \( (A) > \text{Left spread } (B) \) iff \( w_1b_1 > w_2b_2 \).
b. Left spread \( (A) < \text{Left spread } (B) \) iff \( w_1b_1 < w_2b_2 \).
c. Left spread \( (A) = \text{Left spread } (B) \) iff \( w_1b_1 = w_2b_2 \).

Proof: From the condition 1, we have

\[
\mathfrak{R}(A) = \mathfrak{R}(B) \\
\Rightarrow \frac{w_1(a_1 + b_1 + c_1 + d_1)}{4} = \frac{w_2(a_2 + b_2 + c_2 + d_2)}{4} \\
\Rightarrow w_1(a_1 + b_1 + c_1 + d_1) = w_2(a_2 + b_2 + c_2 + d_2) \tag{19}
\]

From the condition 2, we have

\[
\text{mode } (A) = \text{mode } (B) \\
\Rightarrow \frac{w_1(b_1 + c_1)}{2} = \frac{w_2(b_2 + c_2)}{2} \\
\Rightarrow w_1(b_1 + c_1) = w_2(b_2 + c_2) \tag{20}
\]

Also, from the condition 3, we get

\[
\text{divergence } (A) = \text{divergence } (B) \\
\Rightarrow w_1(d_1 - a_1) = w_2(d_2 - a_2) \tag{21}
\]

Solving Eqs. (19), (20) and (21) we get,

\[
w_1a_1 = w_2a_2 \tag{22}
\]
\[ w_i d_i = w_j d_j \quad (23) \]
\[ w_i (b_i - c_i) = w_j (b_j - c_j) \quad (24) \]

Now,

a. Let \( \text{Left spread } (A) > \text{Left spread } (B) \)
   \[ \iff \quad w_i (b_i - a_i) > w_j (b_j - a_j) \]
   \[ \iff \quad w_i b_i > w_j b_j \quad (\because \ w_i a_i = w_j a_j) \]
   Hence, \( \text{Left spread } (A) > \text{Left spread } (B) \iff w_i b_i > w_j b_j . \)

b. Let \( \text{Left spread } (A) < \text{Left spread } (B) \)
   \[ \iff \quad w_i (b_i - a_i) < w_j (b_j - a_i) \]
   \[ \iff \quad w_i b_i < w_j b_j \quad (\because \ w_i a_i = w_j a_i) \]
   Hence, \( \text{Left spread } (A) < \text{Left spread } (B) \iff w_i b_i < w_j b_j . \)

c. Let \( \text{Left spread } (A) = \text{Left spread } (B) \)
   \[ \iff \quad w_i (b_i - a_i) = w_j (b_j - a_i) \]
   \[ \iff \quad w_i b_i = w_j b_j \quad (\because \ w_i a_i = w_j a_i) \]
   Hence, \( \text{Left spread } (A) = \text{Left spread } (B) \iff w_i b_i = w_j b_j . \)

**Corollary 1.** All the results of Proposition 1, are also hold for Right spread.

**Proposition 2.** Let \( A = (a_i, b_i, c_i, d_i; w_i) \) and \( B = (a_j, b_j, c_j, d_j; w_j) \) be two generalized fuzzy numbers such that

1. \( \mathcal{G}(A) = \mathcal{G}(B) , \)
2. \( \text{mode } (A) = \text{mode } (B) , \)
3. \( \text{divergence } (A) = \text{divergence } (B) \)
then

a. \( \text{Left spread } (A) > \text{Left spread } (B) \iff \text{Right spread } (A) > \text{Right spread } (B) . \)

b. \( \text{Left spread } (A) < \text{Left spread } (B) \iff \text{Right spread } (A) < \text{Right spread } (B) . \)

c. \( \text{Left spread } (A) = \text{Left spread } (B) \iff \text{Right spread } (A) = \text{Right spread } (B) . \)

**Proof:** From the Proposition 1, we have
\[ w_i a_i = w_j a_j \]
\[ w_i d_i = w_j d_j \]
\[ w_i (b_i - c_i) = w_j (b_j - c_j) \]
a. \( \text{Left spread } (A) > \text{Left spread } (B) \)
   \[ \iff \quad w_i b_i > w_j b_j \quad \text{(from Proposition 1.)} \]
   \[ \iff \quad w_i c_i < w_j c_j \quad (\because \ w_i (b_i + c_i) = w_j (b_j + c_j) ) \]
   \[ \iff \quad -w_i c_i > -w_j c_j \]
   \[ \iff \quad w_i (d_i - c_i) < w_j (d_j - c_j) \quad (\because \ w_i d_i = w_j d_j) \]
   \[ \iff \quad \text{Right spread } (A) > \text{Right spread } (B) . \]
Similarly b. and c. can be prove easily.

### 2.2 A Complete Strategy to Compare Generalized Fuzzy Numbers

Let \( A = (a_i, b_i, c_i, d_i; w_i) \) and \( B = (a_j, b_j, c_j, d_j; w_j) \) be two generalized fuzzy numbers, then use the following steps to compare \( A \) and \( B . \)
Step 1. Using Definition 3, find the value of $\mathcal{R}(A)$ and $\mathcal{R}(B)$
Case (i): If $\mathcal{R}(A) > \mathcal{R}(B)$ then $A \succ B$.
Case (ii): If $\mathcal{R}(A) < \mathcal{R}(B)$ then $A \prec B$.
Case (iii): If $\mathcal{R}(A) = \mathcal{R}(B)$ then go to Step 2.

Step 2: Using Definition 31, find the value of mode $(A)$ and mode $(B)$
Case (i): If mode $(A) >$ mode $(B)$ then $A \succ B$.
Case (ii): If mode $(A) <$ mode $(B)$ then $A \prec B$.
Case (iii): If mode $(A) =$ mode $(B)$ then go to Step 3.

Step 3: Using Definition 3, find the value of divergence $(A)$ and divergence $(B)$
Case (i): If divergence $(A) >$ divergence $(B)$ then $A \succ B$.
Case (ii): If divergence $(A) <$ divergence $(B)$ then $A \prec B$.
Case (iii): If divergence $(A) =$ divergence $(B)$ then go to Step 4.

Step 4: Using Definition 3, find the value of Left spread $(A)$ and Left spread $(B)$
Case (i): If Left spread $(A) >$ Left spread $(B)$
  i.e., $w_1 b_1 > w_2 b_2$ then $A \succ B$. (From Proposition 1)
Case (ii): If Left spread $(A) <$ Left spread $(B)$
  i.e., $w_1 b_1 < w_2 b_2$ then $A \prec B$. (From Proposition 1)
Case (iii): If divergence $(A) =$ divergence $(B)$
  i.e., $w_1 b_1 = w_2 b_2$ then $A \sim B$. (From Proposition 1)

Step 5: If the values of rank, mode, divergence and left or right spread are equal, then check the value of $w_1$ and $w_2$.
Case (i): If $w_1 > w_2$ then $A \succ B$.
Case (ii): If $w_1 < w_2$ then $A \prec B$.
Case (iii): If $w_1 = w_2$ then $A \sim B$.

Theorem 1. Let $\mathcal{R}$ be a partial ordering relation on $F(X)$ such that $\mathcal{R}(A_i) \neq \mathcal{R}(A_j)$ for $i \neq j$ and $A_i, A_j \in F(X)$ where $F(X)$ be set of generalized fuzzy numbers, then $\mathcal{R}$ form a fuzzy partial ordering relation.

Proof:
(i) Reflexive: Let $A, B$ be any generalized fuzzy number, then $\mathcal{R}(A) \geq \mathcal{R}(A)$
  $\Rightarrow A \succ A \Rightarrow \mathcal{R}$ is reflexive.

(ii) Anti-symmetric: Let $A, B$ be any generalized fuzzy numbers such that $A \succ B$ and
  $B \succ A \Rightarrow \mathcal{R}(A) \geq \mathcal{R}(B)$ and $\mathcal{R}(B) \geq \mathcal{R}(A) \Rightarrow \mathcal{R}(A) = \mathcal{R}(B) \Rightarrow A \sim B$.

(iii) Transitivity: Let $A, B$ and $C$ be any generalized fuzzy numbers such that $A \succ B$
  and $B \succ C \Rightarrow \mathcal{R}(A) \geq \mathcal{R}(B)$ and $\mathcal{R}(B) \geq \mathcal{R}(C)$, Since $\mathcal{R}$ maps each generalized fuzzy number to real line
  $\Rightarrow \mathcal{R}(A) \geq \mathcal{R}(C) \Rightarrow A \succ C$.

Theorem 2. If $A, A'$ and $B$ be any generalized fuzzy numbers such that the ranking values of $A$ and $A'$ are nearer to $B$, then the ranking values of $A$ and $A'$ are not very different.

Or
\[ \forall A, A', B \in F(X) \text{ such that } |\mathcal{R}(A) - \mathcal{R}(B)| \leq \frac{\varepsilon}{2}, \quad |\mathcal{R}(A') - \mathcal{R}(B')| \leq \frac{\varepsilon}{2} \text{ then, } |\mathcal{R}(A) - \mathcal{R}(A')| \leq \varepsilon. \]
Proof: 
\[ |\mathcal{R}(A) - \mathcal{R}(A')| \]
\[ \leq |\mathcal{R}(A) - \mathcal{R}(B) - \mathcal{R}(A') + \mathcal{R}(B)| \]
\[ \leq |\mathcal{R}(A) - \mathcal{R}(B)| + |\mathcal{R}(A') - \mathcal{R}(B)| \]
\[ \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \]
\[ \Rightarrow |\mathcal{R}(A) - \mathcal{R}(A')| \leq \varepsilon \]

5. EXAMPLES AND COMPARISON

Here, we have taken four sets of generalized fuzzy numbers, their ordering is obtained by proposed methods and the results are compared with existing ranking methods

**Set 1**
Let \( A = (-0.2, -0.1, 0.1, 0.2; 0.4) \) and \( B = (-0.1, 0.0, 0.0, 0.1; 0.4) \) be two generalized fuzzy numbers, use the following sets to compare \( A \) and \( B \)
Step 1. Using Definition 3, \( \mathcal{R}(A) = 0 \) and \( \mathcal{R}(B) = 0 \). Since \( \mathcal{R}(A) = \mathcal{R}(B) \) then, go to Step 2.
Step 2. Using Definition 3, mode \( (A) = 0 \) and mode \( (B) = 0 \). Since mode \( (A) = \) mode \( (B) \) then, go to Step 3.
Step 3. Using Definition 3, divergence \( (A) = 0.4 \) and divergence \( (B) = 0.4 \). Since divergence \( (A) > \) divergence \( (B) \Rightarrow A > B \).

**Set 2**
Let \( A = (-0.5, -0.2, 0.3, 0.4; 0.6) \) and \( B = (-0.6, -0.3, 0.4, 0.5; 0.6) \) be two generalized fuzzy numbers, use the following sets to compare \( A \) and \( B \)
Step 1. Using Definition 3, \( \mathcal{R}(A) = 0 \) and \( \mathcal{R}(B) = 0 \). Since \( \mathcal{R}(A) = \mathcal{R}(B) \) then, go to Step 2.
Step 2. Using Definition 3, mode \( (A) = 0 \) and mode \( (B) = 0 \). Since mode \( (A) = \) mode \( (B) \) then, go to Step 3.
Step 3. Using Definition 3, divergence \( (A) = 0.9 \) and divergence \( (B) = 1.1 \). Since divergence \( (A) < \) divergence \( (B) \Rightarrow A < B \).

**Set 3**
Let \( A = (-0.3, -0.2, 0.2, 0.3; 0.7) \) and \( B = (-0.1, -0.1, -0.1, 0.1; 0.7) \) be two generalized fuzzy numbers, use the following sets to compare \( A \) and \( B \)
Step 1. Using Definition 3, \( \mathcal{R}(A) = 0 \) and \( \mathcal{R}(B) = 0 \). Since \( \mathcal{R}(A) = \mathcal{R}(B) \) then, go to Step 2.
Step 2. Using Definition 3, mode \( (A) = 0 \) and mode \( (B) = 0 \). Since mode \( (A) = \) mode \( (B) \) then, go to Step 3.
Step 3. Using Definition 3, divergence \( (A) = 0.6 \) and divergence \( (B) = 0.2 \). Since divergence \( (A) > \) divergence \( (B) \Rightarrow A > B \).
Set 4
Let $A = (-0.4,0.0,0.1,0.3;0.8)$ and $B = (-0.2,0.0,0.1,0.1;0.8)$ be two generalized fuzzy numbers, use the following sets to compare $A$ and $B$.

Step 1. Using Definition 3, $R(A) = 0$ and $R(B) = 0$. Since $R(A) = R(B)$ then, go to Step 2.

Step 2. Using Definition 3, mode $(A) = 0$ and mode $(B) = 0$. Since mode $(A) = mode(B)$ then, go to Step 3.

Step 3. Using Definition 3, divergence $(A) = 0.7$ and divergence $(B) = 0.3$. Since divergence $(A) >$ divergence $(B)$ $\Rightarrow A > B$.

Table 1. Comparison between proposed and existing methods.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Set 1</th>
<th>Set 2</th>
<th>Set 3</th>
<th>Set 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chen and Sanguansat method [6]</td>
<td>$A \sim B$</td>
<td>$A \sim B$</td>
<td>$A \sim B$</td>
<td>$A \sim B$</td>
</tr>
<tr>
<td>Xu et al. [22]</td>
<td>$A \sim B$</td>
<td>$A \sim B$</td>
<td>$A \sim B$</td>
<td>$A \sim B$</td>
</tr>
<tr>
<td>Yagger [23]</td>
<td>$A \sim B$</td>
<td>$A \sim B$</td>
<td>$A \sim B$</td>
<td>$A \sim B$</td>
</tr>
<tr>
<td>Proposed method</td>
<td>$A &gt; B$</td>
<td>$A &lt; B$</td>
<td>$A &gt; B$</td>
<td>$A &gt; B$</td>
</tr>
</tbody>
</table>

Note 1. N.A means method is not applicable.

From Table 1, we can see that for different sets of generalized fuzzy numbers, as shown in Fig. 3, Chen and Sanguansat's method [6], Xu et al. [22] and Yagger [23]) gives the conclusion, $A \sim B$ for all the sets. Whereas, Cheng [7] and Chu and Tsao [8] methods are not applicable. If we look at the structure of each fuzzy number (see Fig 3.), one can easily see that the numbers are not equal. Therefore, the results obtained by existing ranking methods are not reasonable and are not coincide with human intuition. However, with proposed method results obtained are more realistic, reasonable and according with human intuition (see Table 1 and Fig. 3).
In all the existing ranking methods the only criteria, rank is used to compare the fuzzy numbers and generalized fuzzy numbers. We conclude that the rank is not sufficient criteria to find the correct ordering of the generalized fuzzy numbers, so generalized fuzzy numbers needs further evaluation for reasonable ordering. Also, we pointed out the shortcomings of exiting ranking methods. To overcome these shortcomings, we present a new ranking method which considers not only rank, but also mode, divergence and left or right spread criteria of generalized fuzzy numbers. The ranking obtained by proposed method is more realistic, reasonable and according with human intuition. For future research, the proposed method can be applied in real life problems involving decision-making, clustering, optimization, transportation problems, etc.

Fig. 3. Four sets of fuzzy numbers

6. CONCLUSIONS
REFERENCES

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