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Infrastructure Honeycomb Torus with Mutually Independent Hamiltonian Paths

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The infrastructure to support human societies' safety, health, and utility supply can not be negotiated, and to have basic reliability in monitoring tasks, plural or dual surveillance, rooted in biologic senses, is needed. To fit the contemporary wireless communication, MIMO (multi-in multi-out) being incorporated with suggested network prototypes, aimed especially on the naturally bipartite Honeycomb Tori (HT) for cellular communications, to prevent information loss, interference, unexpected changes caused by such as clogged water. The mutually independent Hamiltonian Path (MIHP) property can be used for parallel processing and supporting cipher coding to offer efficiency, integrity, and privacy. Isomorphism between Honeycomb Tori and Generalized Honeycomb Tori (GHT) is utilized; mathematical HT(m) $m \geq 2$ can be presented as GHT (m, $6m$, $3m$). Setting rational configurations, incremental scalability qualifying, it is proved that GHT (m, n, $n/2$) "even m, $n \geq 12$" and "odd $m > 1$, $n \geq 10$," which naturally include full Honeycomb Tori, can have a property dual MIHP.

Keywords: cellular communication, cipher coding, dual surveillance, interference, parallel processing

1. INTRODUCTION

In this century, the idea of the knowledge economy can ubiquitously affect and be affected to comprehensively create a more efficient, fair market and welfare. Present knowledge can be ubiquitously transmitted and interacted with; contemporary societies should openly accommodate more knowledge-based elements in real-time. Hence, the knowledge city [1] with the feature to promote quality of life and develop economy or mobile business through regional interactions, and computing automation has been promoted.

Face to face interactions can be played through flexible means, including video conferencing. People, including the disabled, can get fair chances to be educated and enhance their living to utilize available resources well. Technologies related to computational algorithms and remote communication have played essential roles in better life development, especially from health and security perspectives.

Although traditionally, negative concerns may be aroused when people face the operated surveillance, aggressive surveillance (typically with cameras) has become required more than two decades ago, *e.g.*, in casinos and theme parks. To positively concern terrorism and other unexpected incidents or challenges, the demand of AI (Artificial Intelligence) or biological plural surveillance, such as two (dual) eyes, two ears, and/or the combination

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of different senses – including the olfactory one, apparently essential in fire related disasters.

Radio communication integrity intensely counts on the fault-tolerance and the capabilities countering environmental challenges, including humidity, dynamic occlusion, radio interference, jamming. To help smart urban growth and reasonably inform the government, media, travelers, and residents, enhancing spatial connectivity and flexibility should also be well designed with the ITS. Then the availability of evidence can help illustrate events in time. Furthermore, the systematic maintenance inspection, including the security mattered, is frequently operated through proper order, such as the mathematical Hamiltonian properties of proposed prototypes.

That considering wireless plural surveillance prototype, *e.g.*, on the cellular Honeycomb Torus network, inherited with availability, reliability, maintainability, and information integrity (such as the ability to counter interference and to send dynamic secret information) in spatial information network infrastructure is aimed to support safety and security in the following applications: (1) to counter international terrorism; (2) to concern radio interference due to humid climate and other environmental issues; (3) to promote knowledge economics and/or international business.

2. METHOD

Communication / information networks are usually sketched by graphs in which nodes stand for processors and edges stand for links between processors. It is highlighted that mathematically, scalable performance is beneficial in building up a network prototype, and different dimensional growth need be considered as a whole. The scalability is important for establishing a communication / information sensor-node platform to flexibly support offering the availability for dealing with different environment conditions; the mathematical Hamiltonian order can benefit reliable and efficient maintenance operation (without loss, and with rational efficiency). Therefore, an approach on the reliability in establishing communication / information networks for managing areas, which require quite significant amounts of sensor-nodes for reliable communication / information acquiring, is proposed.

Let $G = (V, E)$ be a graph if V is a finite set and E is a subset of $\{(a, b) \mid (a, b)$ is an unordered pair of $V\}$. A path is denoted by $(x_0, x_1, x_2, \ldots, x_{n\text{-}1})$. A path is called a Hamiltonian path if its nodes are distinct and span V. A cycle is a path of at least three nodes such that the first node is the same as the last node. A cycle is called a Hamiltonian cycle or Hamiltonian if its nodes are distinct except for the first node and the last node, and if they span V.

Assume that m and n are positive integers, where n is even and $m \geq 2$. Let d be any integer such that $(m - d)$ is even. The generalized honeycomb torus [2-4], GHT(m, n, d) is the graph with the node set $\{(i, j) \mid 0 \leq i < m, 0 \leq j < n\}$ such that (i, j) and (k, l) are adjacent if they satisfy one of the following conditions: (1) $i = k$ and $j = l \pm 1 (\text{mod } n)$; (2) $j = l$ and $k = i - 1$ if $i + j$ is even; and (3) $i = 0$, $k = m - 1$, and $l = j + d (\text{mod } n)$ if j is even. GHT(m, $6m$, $3m$) is isomorphic to the bipartite graph, honeycomb torus, HT(m) [3] (Fig. 1). A bipartite graph $G = (V, E)$ is a graph such that $V = A \cup B$ and E is a subset of $\{(a, b) \mid a \in A$ and $b \in B\}$.

Two Hamiltonian paths, $P_1 = (u_1, u_2, \ldots, u_n(G))$ and $P_2 = (v_1, v_2, \ldots, v_n(G))$ of G from u to v are independent if $u = u_1 = v_1$, $v = u_n(G) = v_n(G)$, and $u_i \neq v_i$ for every $1 < i < n(G)$. A set of Hamiltonian paths, $\{P_1, P_2, \ldots, P_k\}$, of G from u to v, are mutually independent if any two distinct paths in the set are independent from u to v [5-9]. It is highlighted that the dual MIHP study is considered for parallel, pact wireless information transmission, diagnosing, and offering additional ciphered information, which is important for offering real-time private information to logistic consigner [7, 8].

Without losing generality, a common start node is assigned in the dual MIHP study of GHT(m, n, n/2), then conditions are classified according to end-nodes' two dimensional separations or scalability, naturally related to m being even or odd. Moreover, varieties such as locations, direction (*e.g.*, toward right or toward left being different yet both along same dimension) need be cared on both graphic and text presentations. On dimension m, the word embedding being generally used, "R-embedding (Red-embedding or R-embd.)" and "*Blue-embedding* (*Blue-embedding* or *B-embd*.) are named and designed in terms of text color and font to highlight location and enhance legibility; the superscripts between two nodes are considered to represent the quantity per one embedding, to highlight potential directional differences, and to clearly process qualifying calculations on basic patter pairs (shown in the Appendix).

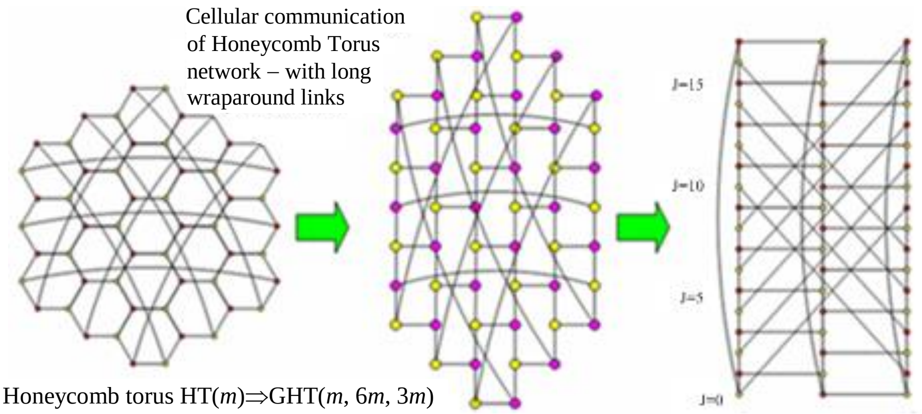


Fig. 1. Graph generation of generalized honeycomb tori from honeycomb tori.

On dimension n, the word extending being generally used, and two kinds of extensions are found; *i.e.*, linear extensions (LE, a series of nodes along one column as a united node) and X extensions (XE, a series of nodes starting and ending at a same column, then two nodes of the adjacent column being connected to make a cyclic extension along a spiral). Special conditions should be considered and highlighted with adaptation marks. It is found that five types can be classified and each type can be classified into three subtypes, which are distinguished from the index, l, the n dimensional distance between endnodes. Proofs are engaged to assure inherent node spaces, including the united node due to linear extension, which generally need to consider issues due to directions.

3. EXPERIMENTS

After some phenomena of dual MIHP have been found, then through rational configurations, incremental scalability qualifying, and proofs, this mathematics related empirical research, from MIHP of Generalized Honeycomb Tori, GHT(m, n, d) $m \geq 2$, $n \geq 4$, $d = n/2$

to Honeycomb Tori, HT(m), is engaged in the next section, and the following figures (Figs. 2-16). Experiment figures and details regarding incremental qualification on embedding location and direction can be viewed in the Appendix.

3.1 Type A – m: even; $l = 0$, $2 \le l \le d - 2$, or d; endnode columns' width-separation odd

(A) Subtype $l = 0$

Both patterns' R-embd., 0K$^{\underline{\underline{8}}8}$1K, 07^{48}17, 15^{44}05, 03^{84}13, can have same contents and direction. Each section can be deemed as a special node, and is conflict-free, as the qualification analysis (refer to Appendix) and Fig. 2. It is noted here to explain the essence of the qualification analysis. That the left toward R-embd., R_l can be used instead of the right toward R-embd. R_r in certain situations. If R_r is utilized on 15^{44}05; counting from right tentatively, $R_r(16) + B(4)$ can be equal to $4 + R_r(4) + B(20)$ or conflicts can happen while "no XE ($X = 0$) and 3 $R_r = 4B + 1$", which consequently, can be (incrementally) avoided if R_l can be utilized for all nodes of both basic patterns in the aforementioned condition.

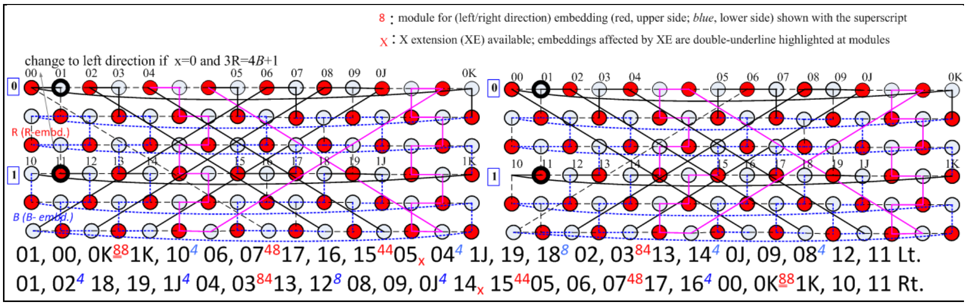


Fig. 2. GHT(m:even, $n \ge 4$, $d = n/2$), MIHP, endnodes at 2 col., separation odd, $l = 0$.

Lt B-embd., 04^{4}1J, 18^{8}02, 14^{4}0J, 08^{4}12 have different direction of Rt B-embd. 1J^{4}04, 02^{4}18, 0J^{4}14, 12^{8}08; yet Lt 18^{8}02 can also have common contents and same direction of Rt 16^{4}00, and Rt 12^{8}08 can have such similarity of Lt 10^{4} 06. Sections of same content, even with different direction, can be deemed as same nodes and can be conflict-free, as the qualification analysis. Cross-sectional Lt-18^{8}02, Rt-12^{8}08 can easily be identified conflict-free with their counter directional Rt-02^{4}18, Lt-08^{4}12 due to enough separation.

XE, in this case, does not cause nodes' location variety. By tracing both XEs, we can find they are conflict-free.

(B) Subtype $2 \le l \le d - 2$

Shown as Fig. 3. Case A. R-embd., Rt-17^{4}07 vs. Lt-17^{8}07, and Rt-15^{4}05 vs. Lt-15^{4}05 have the quality of same direction. Rt-01$^{\underline{\underline{8}}}$11 and Lt-03^{4}13 have same direction and quantity if XE exist as the required ($X > 0$) but have no common contents. Because XE exists, Lt-17^{8}07 is fully separated from Rt-01$^{\underline{\underline{8}}}$11. Lt-03^{4}13 has same direction as that of Rt-XE. Hence, they can have no conflicts.

Both patterns' B-embd., 10^{4}04, 02^{4}16, 00^{4}14, and 06^{4}12 are fully same and can be deemed as same nodes and can be identified conflict-free.

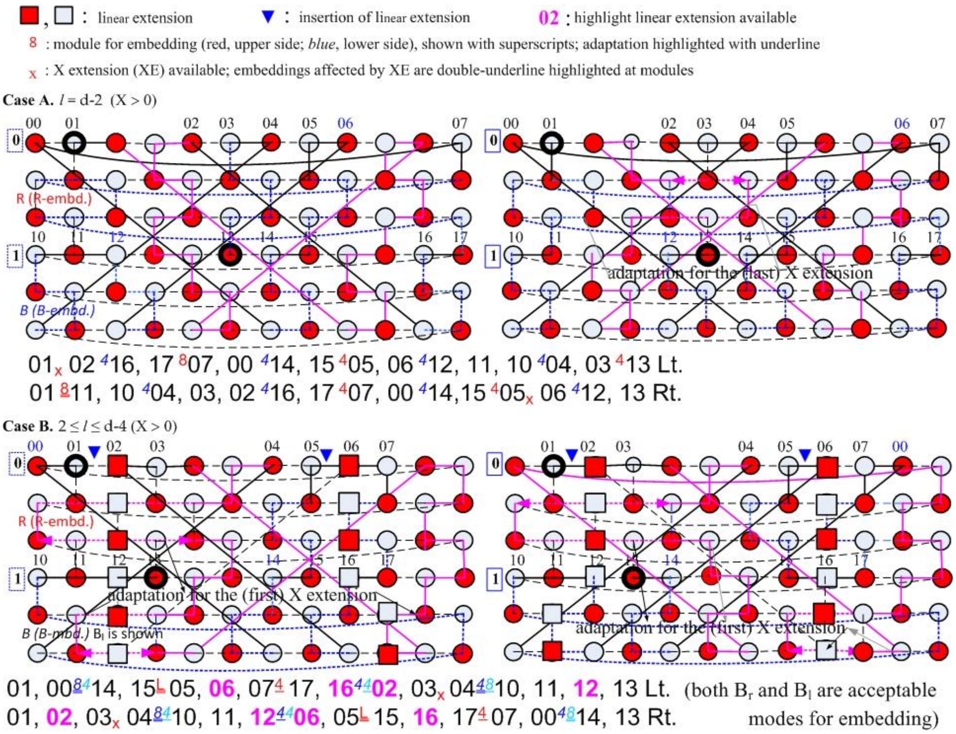


Fig. 3. GHT(m:even, $n \geq 12$, $d = n/2$), MIHP, endnodes at 2 col., separation odd, $2 \leq l \leq d-2$.

XE, in this case, has caused two nodes' location variety, at node 06, 12. By tracing both XEs, we can find they are physically conflict-free.

Shown as Fig. 3. Case B. B_l is chosen with R_l for convenience. All embeddings of both patterns do not cross the other pattern's section, essentially due to the existence of XE. Hence, conflict-free sections can be assured through position qualification analysis. Moreover, B_r can be utilized also, and the dual MIHP can be gotten from different approaches.

Basically, both XEs have same end-nodes and can have same node quantity near end-nodes. So, they keep a regular order without conflicts. In this case, XE can cause two nodes' location variety, 00 and 14. By tracing both XEs, we can find they are conflict-free.

(C) Subtype $l = d$

Shown as Fig. 4, both patterns' R-embd., Lt-01$^{\underline{8}}$11, Rt-03$^{\underline{8}}$13, can have same inner contents if XE does not exist (or X=0) and have same direction. Hence, they can have no conflicts.

Both patterns' B-embd., 12^{4}00, and 02^{4}10 are fully same and can be deemed as same nodes and are conflict-free, as the qualification analysis.

XE, in this case, does not cause nodes' location variety. By tracing both XEs, we can find they are conflict-free.

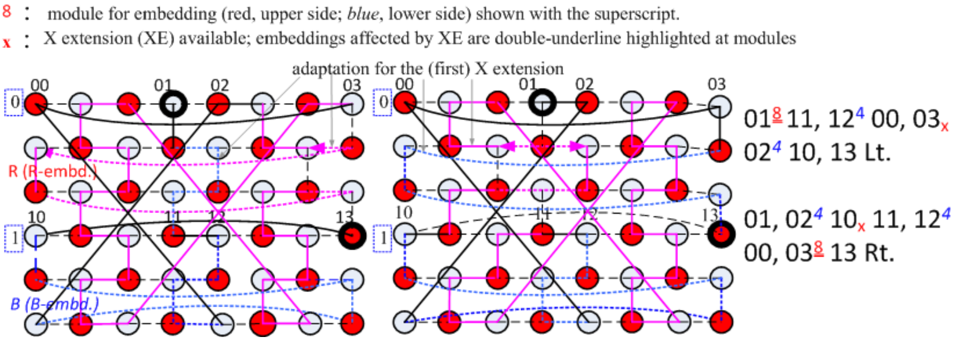


Fig. 4. GHT(m: even, $n \geq 4$, $d = n/2$), MIHP, endnodes at 2 col., separation odd, $l = d$.

3.2 Type B – m: even; l: 1, $3 \leq l \leq d-3$, or $d-1$; endnodes at the same column

(A) Subtype $l = 1$

On Fig. 5. Case A. Lt- **0M**$^{\underline{PP}}$**1M**, **05**$^{\underline{PP}}$**15** have same direction, and their embedding capacity and scopes can compatibly accommodate all embeddings, without conflicts, within Rt "01^{44}11, 13^{44}03, **05**$^{\underline{48}}$**15**," and "1K^{4}0K, 09^{84}19," respectively. Rt-0O$^{\underline{84}}$1O has same direction as Lt two embedding sections', and consequently can maintain their original relative order without conflicts.

On Fig. 5. Case B. Lt-1K^{L}0K, 15^{L}05 have same direction, and their embedding capacity and scopes can compatibly accommodate all embeddings, without conflicts, within Rt "01^{44}11, 13^{44}03, 05^{4}15," and "09^{8}19, 1K^{4}0K," respectively. They can maintain their original relative order without conflicts.

(B) Subtype $3 \leq l \leq d-3$

On Fig. 6, Case A. Lt-17$^{\underline{4}}$07, **09**^{4}**19** together can have same contents of Rt-07^{4}17, **19** ^{4}**09** although single section cannot; enough node quantity due to Rt-01 $^{\underline{8}}$11, 05^{4}15 can support them conflict-free. Other embedding sections of both patterns, *i.e.*, Lt-03^{4}13, 05^{4}15, 0K$^{\underline{8}}$1K and Rt-01$^{\underline{8}}$11, 05^{4}15, 0K^{4}1K have same direction so that they can maintain conflict-free, *i.e.*, by keep leading / lagging.

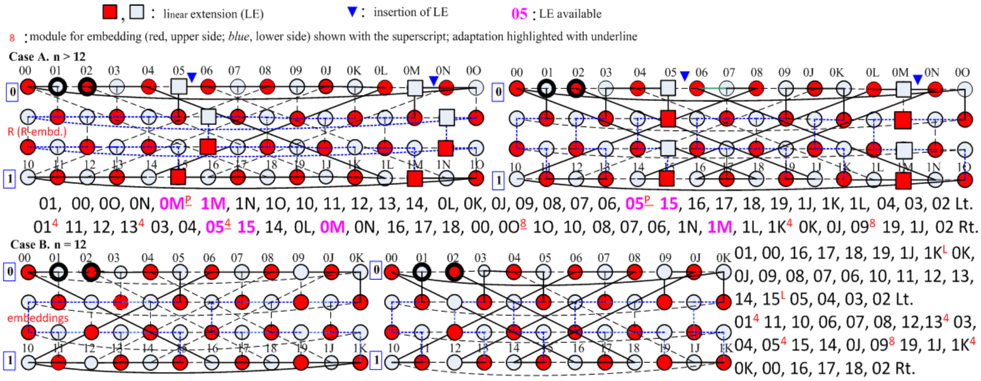


Fig. 5. GHT(m: odd, $n \geq 6$, $d = n/2$), MIHP, endnodes at 2 col., l=0 (Ł=12+a pair of LE).

XE, in this case, does not cause nodes’ location variety although the embedding within Lt-0K$^{\underline{\underline{8}}}$1K can be partly joined in XE. By tracing both XEs, we can find they are physically conflict-free.

On Fig. 6, Case B. “$E \neq 0$” & “E mod 4 = 0” can avoid conflicts, including node 0k, as the qualification analysis.

The embedding quantity can work as a buffer. Lt-03^{4}13, 1K$^{\underline{8}}$0K, “15$^{\underline{8}}$**05**, 09^{4}19” can balance Rt-01^{4}11, 19$^{\underline{8}}$09, “03$^{\underline{4}}$13, **15**^{8}**05**”, respectively. Node conflicts of each balanced block can be analyzed independently. We can find the above three balanced blocks are conflict-free and can maintain original order.

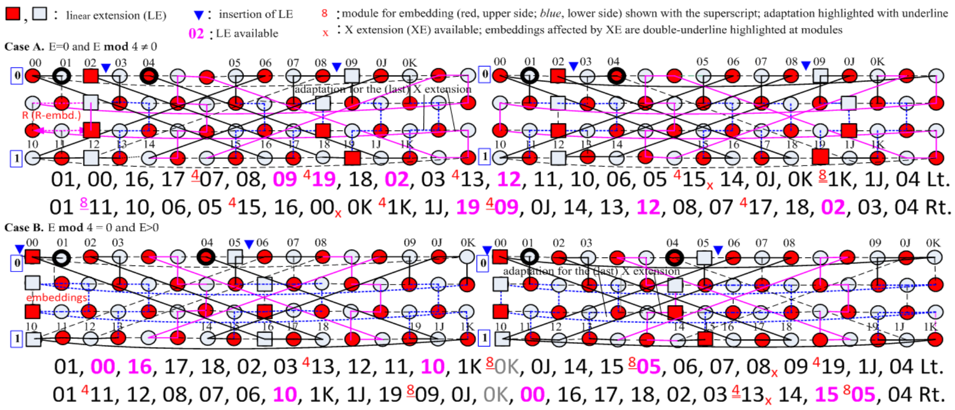


Fig. 6. GHT(m: odd, $n \geq 12$, $d = n/2$), MIHP, endnodes at the same col., $3 \leq l \leq d-3$.

XE, in this case, does not cause nodes’ location variety although the embedding within Rt-03$^{\underline{\underline{4}}}$13 can be partly joined in XE. By tracing both XEs, we can find they are physically conflict-free.

(C) Subtype $l = d - 1$

On Fig. 7, Lt- 1J$^{\underline{L}}$04 or Rt-**18**$^{\underline{L}}$02 similarly can have half quantity of a full embedding, *i.e.*, 12+2E, which can just balance the quantity of “Rt-12$^{\underline{8}}$**08**, 00 1^{4}6” or “Lt-0J^{8}14, **18**$^{\underline{4}}$02.” Such balance does not cause conflicts because the compared embeddings either can have same direction or can have no common contents. Consequently, both patterns can maintain their original relative order without conflicts.

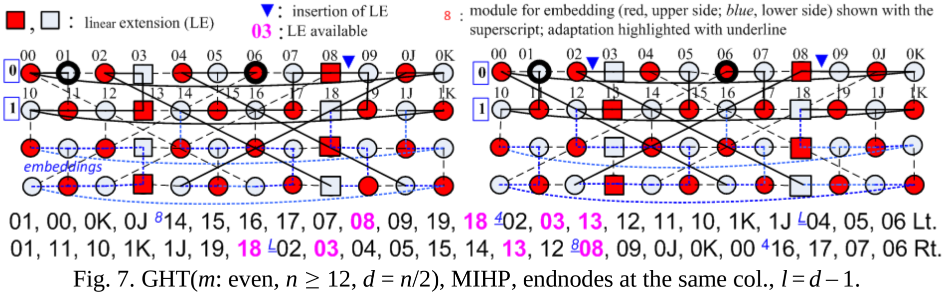
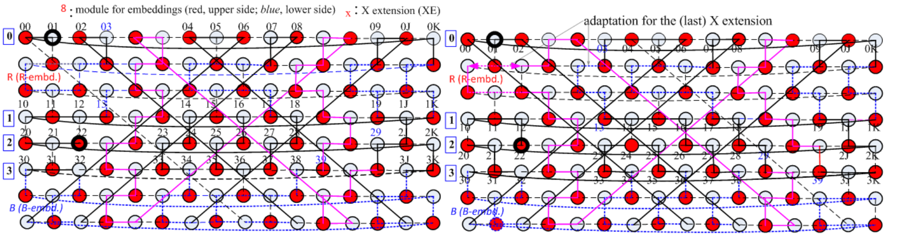


Fig. 7. GHT(m: even, $n \geq 12$, $d = n/2$), MIHP, endnodes at the same col., $l = d-1$.

3.3 Type C – m: even; l: 1, $3 \le l \le d-3$, or $d-1$; endnode columns' width-separation even

(A) Subtype $l = 1$

Both patterns of R embeddings, on Fig. 8, have four sections. Except Lt-$13^{8}03$ and Rt-$13^{\underline{8}}03$ each section has same corresponding section, and is conflict-free as the qualification analysis. The difference between Lt-$13^{8}03$ and Rt-$13^{\underline{8}}03$ is the latter can be separated into two parts and mixed with XE although by keeping same direction and order, they are conflict-free.



01, 00, $0K^{4}$ 1K, 10, 11, 12, 13^{8} 03, 02^{8} 38, 39_{x} 29, 2J, 1J, 19^{4} 09, $0J^{4}$ 34, 33, 23, 24, 14, 15, 16, 26, 25, 35, 36, 37, 27, 28, 18, 17^{8} 07, 08^{4} 32, 31, 30^{4} 06, 05, 04^{4} 3J, 3K, 2K, 20, 21, 22 Lt.
01, 02^{4} 38, 37, 27, 26, 16, 15, 14, $13^{\underline{8}}$ 03, 04, 05, 06^{8} 30, 3K, 3J, 39_{x} 29, 28, 18, 17^{8} 07, 08^{4} 32, 31, 21, 20, 2K, 2J, 1J, 19^{4} 09, $0J^{4}$ 34, 33, 23, 24, 25, 35, 36^{4} 00, $0K^{4}$ 1K, 10, 11, 12, 22 Rt.

Fig. 8. GHT(m: even, $n \ge 12$, $d = n/2$), MIHP, endnodes at 2 col., separation even, $l = 1$.

B embeddings, $0J^{4}34$ and $08^{4}32$ of both patterns are fully same and are conflict-free as the qualification analysis. Lt-$30^{4}06$ has different direction of and less quantity than Rt-$06^{8}30$, but can be assisted with other B embeddings as a buffer and can keep leading. Rt-$06^{8}30$, except the same quantity within Lt-$30^{4}06$, has same direction and quantity as Lt-$04^{4}3J$, which can keep leading.

Rt-$02^{4}38$ has same direction of but less quantity than Lt-$02^{8}38$. Rt-$02^{4}38$ can naturally keep lagging. Rt-$36^{4}00$, has same quantity within Lt-$02^{8}38$, but has different direction; yet it can be assisted with other B embeddings as a buffer and can keep leading.

XE, in this case, has caused four nodes' location variety, at node 03, 13, 29, 39. By tracing both XEs, we can find they are physically conflict-free.

(B) Subtype $3 \le l \le d-3$

On Fig. 9, Case A. R-embd. $15^{\underline{44}}05$ of both patterns are fully same and can be considered as special node without conflicts, as the qualification analysis. R_l, used with B_r, Lt-$17^{44}07$ has same quantity as Rt-$07^{48}17$ but has different direction, and can keep lagging. Rt-**0K**^{84}**1K** can aaacoomodate the embedding content of Lt-$09^{44}19$ and has same direction. They may have conflicts if the B/R embedding is small and direction adaptation neglected, so that Lt-$09^{\underline{44}}19$ can become lagging, refer to the qualification analysis. Rt-**0K**$^{8\underline{4}}$**1K** can also accommodate the embedding content of Lt-**1K**$^{\underline{4}8}$**0K**, with different direction but they have no conflicts due to enough embedding buffers. Lt-$03^{\underline{84}}13$ has same direction and common embedding contents of Rt-$01^{\underline{44}}11$, $03^{\underline{44}}13$, they apparently have no conflicts and Rt-$01^{\underline{44}}11$ keeps lagging, Rt-$03^{\underline{44}}13$ keeps leading.

R_r, only is used with B_l and for small embeddings. Lt-**1K**$^{\underline{4}8}$**0K** has common content of Rt-$01^{\underline{4}4}11$, **0K**$^{8\underline{4}}$**1K** and can keep leading with enough embeddings. Rt-$07^{48}17$ has common content of Lt-$17^{44}07$, $09^{\underline{44}}19$ and can keep leading with enough embeddings. Rt-$03^{\underline{44}}13$ can have fully same attributes as Lt-$03^{\underline{84}}13$ with no conflicts naturally.

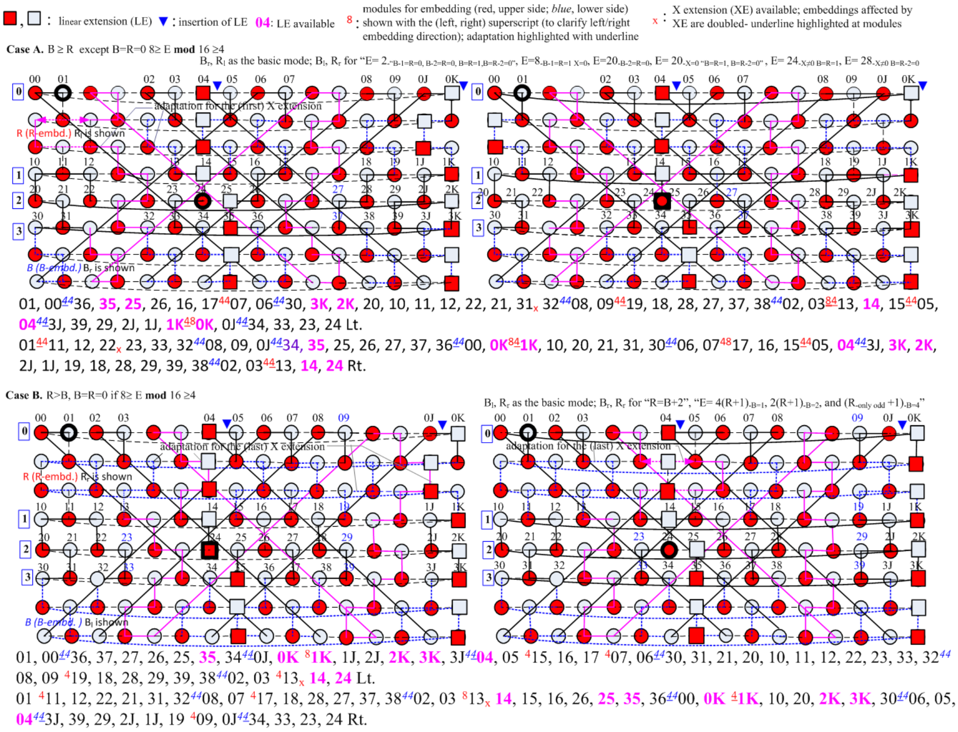


Fig. 9. GHT(m: even, $n \geq 12$, $d = n/2$), MIHP, endnodes at 2 col., separation even, $3 \leq l \leq d-3$.

B-embd., all or six embedding sections of both patterns have same quantity in both directions, But 32^{44}08, 0J^{44}34, **04**^{44}3J, 38^{44}02 of both patterns can also get common direction, the other two sections have different direction. All of them are conflict-free as the qualification analysis.

XE, in this case, has caused two nodes' location variety, at nodes 27, 37. By tracing both XEs, we can find they are physically conflict-free.

On Fig. 9, Case B. R_r, except in "R=2, B=0", is used for R embedding. Lt-**0K**^{48}**1K**, has common embedding contents of Rt-**0K**^{44}**1K**, 01^{44}11 and has same direction so that they can be conflict-free. The LE within Rt-**0K**^{44}**1K** can have common LE quantity of Lt 09^{44}19 if X=0 or the adapted quantity of XE, and they have same direction so that they can be conflict-free. Rt-03^{48}13 has common embedding contents of Lt-05^{44}15, 03^{84}13 if X=0 or the adapted quantity of XE, and has same direction so that they can be conflict-free. Lt-09^{44}19, except LE, has same contents of Rt-19^{44}09, the latter can keep leading due to enough embedding assistance. Lt-17^{44}07 has same contents of Rt-07^{84}17, the former can keep leading due to enough embedding assistance. Through the qualification analysis, R_l can be applied for "R=2, B=0". Due to same direction and/or enough buffers, they are conflict-free.

B-embd., all or six embedding sections of both patterns have same quantity in both directions, but 32^{44}08, 38^{44}02 of both patterns can also get common direction, the other four sections have different direction. All of them are conflict-free as the qualification analysis.

XE, in this case, has caused two nodes' location variety, at nodes 23, 33, 09, 19, 29, 39. By tracing both XEs, we can find they are physically conflict-free.

(C) Subtype l = d-1

On Fig. 10, only Rt-**12**$^{\underline{4L}}$22 has different embedding direction in R embedding. Its R_l has no common contents of Lt-26$^{\underline{88}}$16, and R_r is used for B being a multiple of "4R+2". Consequently, the R embedding can have no conflicts. Only Rt-**25**^{4}35 has different embedding direction in B embedding. Lt-**35**$^{\underline{4}}$**25** has common contents, though Rt-33$^{\underline{8}}$ 23 can prevent their conflicts.

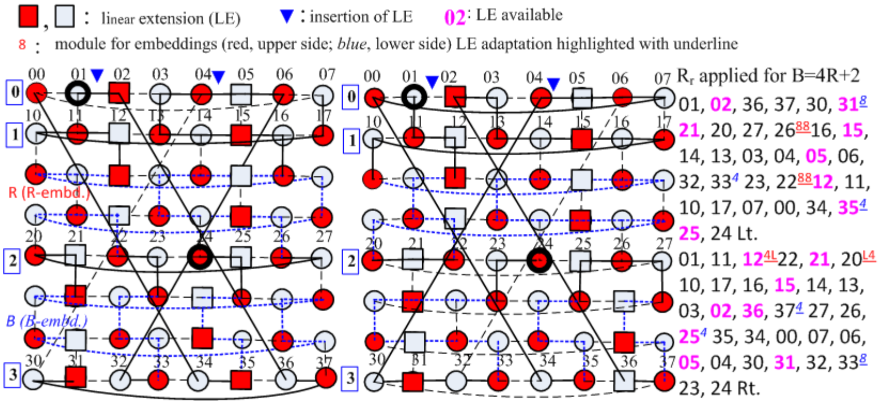


Fig. 10. GHT(m: even, $n \geq 8$, $d = n/2$), MIHP, endnodes at 2 col., separation even, $l = d - 1$.

3.4 Type D – m: odd; l: 0, $2 \leq l \leq d-3$, or $d-1$; endnodes at different columns

(A) Subtype $l = 0$

On Fig. 11, 05$^{\underline{L}}$15 of both patterns can considered as a special node, and it is conflict-free. Lt-**12**$^{\underline{84}}$**22** has same quantity and direction as Rt-**10**$^{\underline{84}}$**20**. Such quantity can benefit keeping original order, and same direction helps they have no conflicts. Similarly, Lt-**20**$^{\underline{48}}$**10** and Rt-24$^{\underline{48}}$14 have no conflicts.

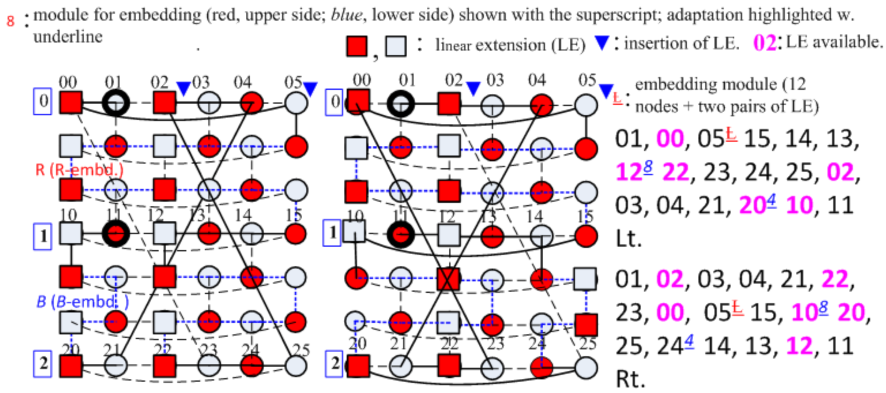
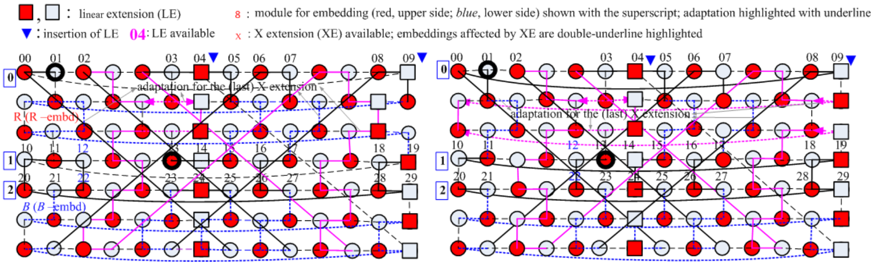


Fig. 11. GHT(m: odd, $n \geq 6$, $d = n/2$), MIHP, endnodes at 2 col., $l = 0$.

(B) Subtype $2 \le l \le d-3$

On Fig. 12, Lt-**09**$^{\underline{L}}$**19**, has common contents of Rt-01$^{\underline{8}}$11; both sections have same direction and may separate some contents to XE, which has same direction as the aforementioned two sections. Lt-07$^{\underline{4}}$17, 05^{4}15 have same direction ad Rt-07$^{\underline{8}}$17, 05^{4}15; hence, they have no conflicts. Lt-07$^{\underline{4}}$17, Rt-07$^{\underline{8}}$17 may separate some contents to XE ($X>0$), then they have same contents as a special node without conflicts.



01, 00$^{\underline{4}}$ 25, **24**, 23^{4} 08, **09** $^{\underline{L}}$**19**, 18, 28, **29**, 20, 10, 11, 12, 22, 21$^{\underline{8}}$ 06, 07 $^{\underline{4}}$17, 16, 26, 27^{4} 02$_{x}$ 03, **04**, 05 ^{4}15, **14**, 13 Lt.
01 $^{\underline{8}}$11, 10, **19**, 18, 28, **29**, 20, 21$^{\underline{8}}$ 06, 07 $^{\underline{8}}$17, 16, 26, 27^{4} 02$_{x}$ 03, **04**, 05 ^{4}15, **14**, **24**, 25$^{\underline{4}}$ 00, **09**, 08^{4} 23, 22, 12, 13 Rt.

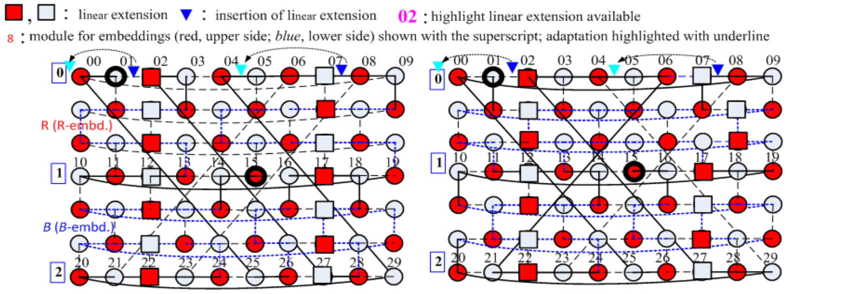
Fig. 12. GHT(m: odd, $n \ge 10$, $d = n/2$), MIHP, endnodes at 2 col., $2 \le l \le d-3$.

Lt-21$^{\underline{8}}$06 and Rt-21$^{\underline{8}}$06 have same direction, contents as a special node without conflicts. Similarly, Lt-27^{4}02 / 21$^{\underline{8}}$06 and Rt-27^{4}02 / 21$^{\underline{8}}$06 have same direction, contents as a special node without conflicts. Though the direction is different, "Lt-00$^{\underline{4}}$25 and Rt-25$^{\underline{4}}$00" still can work as a special node without conflicts.

XE, in this case, has caused two nodes' location variety, at nodes 12, 22. By tracing both XEs, we can find they are physically conflict-free.

(C) Subtype $l = d-1$

On Fig. 13, case A (the configuration for the first step, which is capable for most situations – except for "$E = 2$ and no R embedding). Rt-**14**$^{\underline{8}}$**24** can accommodate the same quantity of the sum of the embedding of Lt-26$^{\underline{4}}$ 16, 18^{4}28, but their contents are irrelevant so that no conflicts or impacts can cause. Similarly, Lt-24$^{\underline{L}}$**14** can accommodate the same quantity of the sum of the embedding of Rt-**20**$^{\underline{8}}$**10**, 26^{4}16, they have common contents yet have same direction so that no conflicts can cause.



01, 00, 25, 26^{4} 16, **17**, 18$^{\underline{4}}$ 28, **27**, **02**, 03$^{\underline{8}}$ 13, **12**, 11, 10, 19$^{\underline{L}}$ 09, 08, **07**, 06, 05, 04, 29, 20, 21, **22**, 23, 24$^{\underline{L}}$ 14, 15 Lt.
01$^{\underline{8}}$ 11, **12**, 13, 14$^{\underline{8}}$ 24, 23, **22**, 21, 06, 05, 04, 03, **02**, **27**, 28, 29, 20$^{\underline{8}}$ 10, 19, 18, **17**$^{\underline{L}}$ **07**, 08, 09, 00, 25, 26^{4} 16, 15 Rt.
↑R≠0 and "R=0, E≠2"; ↓ R=0, E=2 (adapting LE location shown as arrows' direction)
01, **00**, **25**, 26$^{\underline{4}}$ 16, 17, 18^{4} 28, 27, 02, 03, 13, 12, 11, **10**, 19, 09, 08, 07, 06, 05, **04**, 29, **20**, 21, 22, 23, 24$^{\underline{L}}$ **14**, 15 Lt.
01, 11, 12, 13, **14**$^{\underline{8}}$ **24**, 23, 22, 21, 06, 05, **04**, 03, 02, 27, 28, 29, **20**$^{\underline{8}}$ **10**, 19, 18, 17, 07, 08, 09, **00**, 25, 26^{4} 16, 15 Rt.

Fig. 13. GHT(m: odd, $n \ge 10$, $d = n/2$), MIHP, endnodes at 2 col., $l = d-1$.

On Fig. 13, case B (the configuration for the second step, which is capable for "$E = 2$ and no R embedding" or "R embedding $\neq 0$"; the insertion location for the scalability concern is shown with curved arrows, yet related linear extension is presented only by texts). Rt-**14**$^{\underline{8}}$**24** can accommodate the same quantity of the sum of the embedding of Lt-26^{4}16, 18$^{\underline{4}}$ 28, but their contents are irrelevant so that no conflicts or impacts can cause. Similarly, Lt-24$^{\underline{L}}$**14** can accommodate the same quantity of the sum of the embedding of Rt-**20**$^{\underline{8}}$**10**, 26^{4}16, they have common contents yet have same direction so that no conflicts can cause.

Lt-03$^{\underline{8L}}$13 has same embedding quantity as Rt-01$^{\underline{8L}}$11, Lt-19$^{\underline{L8}}$09 has same embedding quantity as Rt-**17**$^{\underline{L8}}$**07** has, and they have same direction so that no conflicts or impacts can cause.

3.5 Type E – $m \geq 3$: odd; l: 1, $3 \leq l \leq d-2$, or d; two columns at the same column

(A) Subtype $l = 1$

On Fig. 14, R and B embeddings are incorporated alternating pairs. Especially being in the first embedding, Lt-07$^{\underline{8L}}$17 cannot interfere with Rt-19 $^{\underline{4}}$09 due to enough embedding assistance and original separation. Rt-13$^{\underline{48}}$03 and Lt-13$^{\underline{48}}$03 can be deemed as a special node without conflicts. Lt- 27$^{\underline{L8}}$02 cannot interfere with Rt-04$^{\underline{84}}$29 due to enough embedding assistance and original separation. Especially being in the third embedding, Lt-06$^{\underline{8L}}$21 cannot interfere with Rt-23$^{\underline{44}}$08 due to enough embedding assistance and original separation.

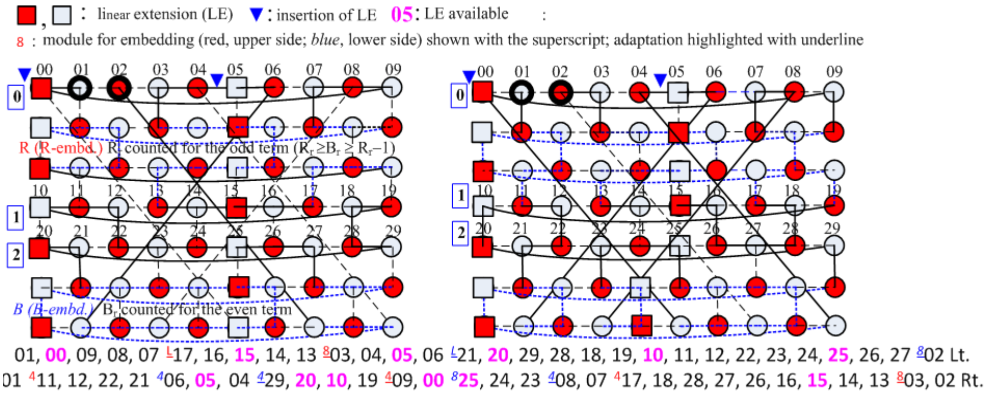


Fig. 14. GHT(m: odd, $n \geq 10$, $d = n/2$), MIHP, endnodes at one col., $l = 1$.

(B) Subtype $2 \leq l \leq d-3$

On Fig. 15, Rt-**00**$^{\underline{4}}$**25** has the embedding contents of Lt-**00**^{4}**25** and the embedding LE of Lt-27$^{\underline{4}}$02. Due to enough separation ("toward-right" mode related) and embedding, they don't have conflicts. Lt-23$^{\underline{8}}$08 can fully contain without conflicts with the embedding contents of same-direction Rt-23^{4}08 Related to XE existence, Lt-23$^{\underline{8}}$08 can smoothly contain a part of same-direction Rt-21$^{\underline{8}}$06, or be adapted into XE with same direction. 27$^{\underline{4}}$02 of both patterns is a special node without conflicts. Lt-29^{4}04 is compatible with the same contents within Rt-21$^{\underline{8}}$06 due to same direction and enough separation. The "toward-right" mode is applied only once; by checking cross-sectioned sections, *i.e.*, Lt-**00**$^{\underline{44}}$**25**, 29$^{\underline{48}}$04 and Rt-27$^{\underline{48}}$02, 23$^{\underline{44}}$08, we can find they have no conflicts.

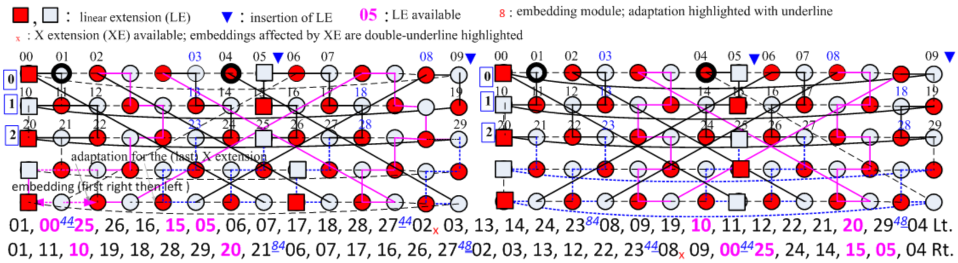


Fig. 15. GHT(m: odd, $n \geq 10$, $d = n/2$), MIHP, endnodes at 2 col., $2 \leq l \leq d-3$.

XE, in this case, has caused six nodes' location variety, at nodes 08, 18, 28, 03, 13, 23. By tracing both XEs, we can find they are physically conflict-free.

(C) Subtype $l = d$

On Fig. 16, the embedding, Lt-05$^{\underline{L}}$15 has same embedding contents and direction as Rt-**03**$^{\underline{L}}$**13**. Their sequential order has some differences that can help them conflict-free.

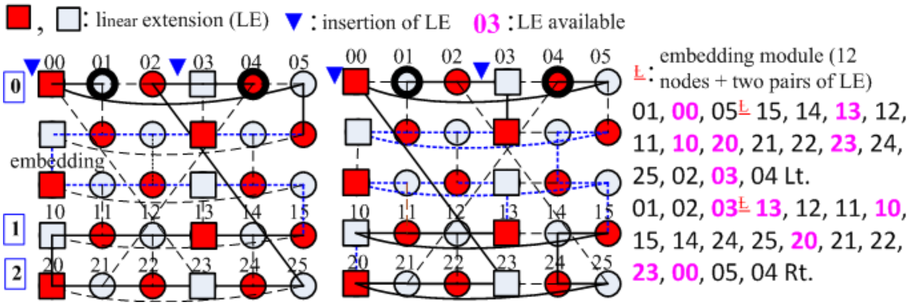


Fig. 16. GHT(m: odd, $n \geq 6$, $d = n/2$), MIHP, endnodes at the same col., $l = d$.

In summary, some critical, but not absolutely effective, mathematical or graph means in these dual MIHP studies are as the following.

(1) same direction
(2) blocking
(3) non-divisibility
(4) reasonable linear extension
(5) reasonable X-helix extension

4. CONCLUSION

The contemporary economy significantly counts on globalization activities. Openness or incorporation participation in knowledge-based economy is getting more attention. A just-in-time economy can push the strategy of using computational transportation facilities, which naturally can benefit tourism and business if security-information services can be synergistically recognized.

Just as mankind use two eyes for seeing, two ears for hearing, two nostrils for the smell to well sense and communicate the changing environments and related information, plural surveillance-information networking is suggested, especially in this age that wireless telecommunications have been prevailed.

The mutually Independent Hamiltonian Paths property (MIHP) can be utilized for parallel analyzing interference and supporting cipher coding to offer privacy. Similarly, Honeycomb tori, can offer the alternative pattern *i.e.*, cellular communication. It is found that the honeycomb tori, HT(m) $m \geq 2$, can have dual MIHP, or GHT(m, n, n/2), the specific generalized honeycomb tori can have dual MIHP property if "even m, $n \geq 12$" or "odd $m > 1$, $n \geq 10$."

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