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A Spherical Fuzzy Entropy Measure with its Application in the Selection of the Best Market Segment*

ABDUL HASEEB GANIE[1+], SAMIRAH ALZAHRANI[2], AMIRA M SEIF[3]
AND MOHAMMAD M KHALAF[4]
[1]*Department of Mathematics, NIT Warangal, Warangal,Telangana 506004, India*
[2]*Department of Mathematics and Statistics, College of Science, Taif University, Taif, Saudi Arabia*
[3]*Future High Institute of Engineering in Fayoum, Egypt*
[4]*Department of Mathematics, Faculty of Science, Tobrouk University, Libya*
E-mail: ahg110605@gmail.com (Abdul Haseeb Ganie), samar.alz@tu.edu.sa (Samirah Alzahrani), amira.sief6923@fief.edu.eg (Amira M Sief), khalfmohammed2003@yahoo.com (Mohammad M Khalaf)

Fuzzy logic is an effective approach to handle the inherent unpredictability and ambiguity in decision-making. One of the latest advancements in this field is the concept of spherical fuzzy sets. By ensuring that the sum of squared values of membership, non-membership, and hesitation degrees remains between 0 and 1, and defining each degree within [0, 1], the hesitation of decision-makers regarding an attribute can be captured more comprehensively. In this study, we propose a spherical fuzzy entropy measurement and demonstrate its ability to meet key axiomatic requirements. We also compare it with existing spherical fuzzy information metrics in areas such as ambiguity computation, linguistic hedges, and attribute weight calculation. Utilizing the proposed entropy metric, we introduce the Complex Proportional Assessment (COPRAS) method for spherical fuzzy sets and provide a numerical example focused on selecting the optimal market sector.

Keywords: Fuzzy set, spherical fuzzy set, ambiguity, linguistic hedges, multi-attribute decision-making

1. INTRODUCTION

Zadeh [1] introduced the idea of fuzzy sets (FSs) to handle the ambiguous situations. When FSs were discovered to be useful in "computer sciences," "communications," "intellectual sciences," "decision sciences," and "engineering," researchers grew interested in them.

As a generalisation of fuzzy sets (FSs), Atanassov [2] introduced the idea of intuitionistic fuzzy sets (IFSs). Every element in an intuitionistic fuzzy set (IFS) has a level of membership (σ) and a level of non-membership (τ) such that $\sigma + \tau \le 1, 0 \le \sigma, \tau \le 1$. Nevertheless, IFSs cannot manage the cases where $\sigma + \tau > 1$because of the restriction $\sigma + \tau \le 1$. Thus, to give more freedom to decision makers, Yager [3] introduced the idea of Pythagorean fuzzy sets (PYFSs). Every element in a Pythagorean fuzzy set (PYFS) has a level of membership (σ) and a level of non-membership (τ) such that $\sigma^2 + \tau^2 \le 1, 0 \le \sigma, \tau \le 1$.

The FSs and IFSs are unable to handle situations that involve the concept of neutrality. For example human voting, machine vision, feature selection, medical diagnosis, etc. To answer these issues, the idea of a picture fuzzy set (PIFS) was framed by Cuong and Kreinvoch [4]. A PIFS gives each of its elements a satisfaction level (σ), a non-satisfaction level (τ), and a neutrality level (φ) such that $\sigma + \tau + \varphi \le 1, 0 \le \sigma, \tau, \varphi \le 1$. This new concept is very close to human nature than the existing ones and is currently a trending research area now because of its applicability in image processing, decision-making, classification, etc.

The PIFSs suggested by Cuong and Kreinvoch [4] are more efficient and reliable than the FSs and IFSs but due to the constraint $\sigma + \tau + \varphi \leq 1$, their scope is limited. So, the PIFSs were generalized and a new concept known as spherical fuzzy sets (SFSs) was introduced by Mahmood et al. [5] and Ashraf et al. [6]. An SFS gives each of its elements a satisfaction level (σ), a non-satisfaction level (τ), and a neutrality level (φ) such that $\sigma^2 + \tau^2 + \varphi^2 \leq 1, 0 \leq \sigma, \tau, \varphi \leq 1$. This means that FSs, IFSs, PYFSs, and PIFSs are a part of the space of SFSs and so SFSs are more robust and effective than all of these types of FSs. Some basic operations of SFSs were given by Mahmood et al. [5]. Kutlu and Kahraman [7] extended the TOPSIS method to the SF environment. Ashraf and Abdullah [8] studied various SF aggregation operators along with their application in decision-making.

A fascinating question is how to calculate FS's level of uncertainty. The quantity of information produced by a random process is measured by the entropy. More information (uncertainty) in the process is indicated by a higher value of entropy. Shannon [9] defines entropy as a theoretical assessment of the inherent uncertainty in information, which can be divided into three categories: fuzzy, non-specific, and contradictory. Barukab et al. [10] proposed an SF entropy metric with its utility in group decision-making. Aydogdu and Gul [11] introduced an entropy metric for SFSs and utilized it for computing attribute weights in the SF-WASPAS method. Li et al. [12] suggested a knowledge-based SF entropy metric with its applicability in the determination of expert weights in decision-making problems. However, all of the available SF information metrics give unreasonable results in many situations. Therefore, a new SF information metric is desirable.

The main motivating factors for this study are as:

(1) The existing SF entropy metric due to Barukab et al. [10] gives “0.5” as the ambiguity content for all those spherical fuzzy numbers in which membership and non-membership are equal i.e. $\sigma = \tau$. This is not reasonable for computing the amount of uncertainty.

(2) The SF entropy metric due to Aydogdu and Gul [11] gives $E(C_1) = 1$, when $C_1 = (a, a, 0.5), 0 \leq a \leq 1$. This means that for different values of a, we get different SFSs but all of them have entropy equal to “1”, which is totally irrational.

(3) All of the existing SF entropy metrics [10–12] lead to unreasonable results in the computation of ambiguity of different SFSs and also in the attribute weight computation.

(4) All of the available SF entropy metrics [10–12] are unable to handle the linguistic hedges properly.

So, because of the above factors, we introduce a novel SF entropy measure in this paper. The following are the study's main contributions:

(1) We offer a novel SF entropy metric based on all four membership levels and establish its validity.

(2) We contrast the offered SF entropy metric with all of the available SF entropy metrics through various examples related to ambiguity computation and attribute weight computation in the SF environment.

(3) We establish its superiority over the available SF entropy metrics through linguistic hedges.
(4) We present the novel COPRAS method in the SF area using the proposed SF entropy metric and illustrated it with the help of a decision-making problem related to the identification of the best market segment.

The manuscript is organized as Section 2 is preliminary. A novel SF entropy metric with its properties is discussed in Section 3. In Section 4, it is illustrated how the recommended SF entropy function compares numerically to the available SF information measurements in various ways. Section 5 introduces the novel COPRAS method in the SF environment based on the developed SF entropy metric. A decision-making problem concerning the selection of the best market sector is also discussed in Section 5. Finally, the conclusion along with future studies are discussed in Section 6.

2. PRELIMINARY

Here $SFS(B)$ denote the collection of all SFSs in the universe $B = \{b_1, b_2, \dots, b_p\}$.

Definition 1 [1] A fuzzy set C_1 in B is given by

$$C_1 = \left\{\left(b_t, \sigma_{C_1}(b_t)\right), b_t \in B\right\},$$

where $0 \le \sigma_{C_1}(b_t) \le 1$ is the grade of satisfaction of $b_t \in B$ in the set C_1.

Definition 2 [4] A picture fuzzy set C_1 in B is given by

$$C_1 = \left\{\left(b_t, \sigma_{C_1}(b_t), \tau_{C_1}(b_t), \varphi_{C_1}(b_t)\right), b_t \in B\right\},$$

where $0 \le \sigma_{C_1}(b_t) \le 1$, $0 \le \tau_{C_1}(b_t) \le 1$, and $0 \le \varphi_{C_1}(b_t) \le 1$ are the grades of satisfaction, non-satisfaction, and neutrality respectively of $b_t \in B$ in the set C_1 such that $0 \le \sigma_{C_1}(b_t) + \tau_{C_1}(b_t) + \varphi_{C_1}(b_t) \le 1$. Also, $\omega_{C_1}(b_t) = 1 - \sigma_{C_1}(b_t) - \tau_{C_1}(b_t) - \varphi_{C_1}(b_t)$ is the refusal degree for the element $b_t \in B$ in the set C_1.

Definition 3 [5] A spherical fuzzy set C_1 in B is given by

$$C_1 = \left\{\left(b_t, \sigma_{C_1}(b_t), \tau_{C_1}(b_t), \varphi_{C_1}(b_t)\right), b_t \in B\right\},$$

where $0 \le \sigma_{C_1}(b_t) \le 1$, $0 \le \tau_{C_1}(b_t) \le 1$, and $0 \le \varphi_{C_1}(b_t) \le 1$ are the grades of satisfaction, non-satisfaction, and neutrality respectively of $b_t \in B$ in the set C_1 such that $0 \le \sigma^2_{C_1}(b_t) + \tau^2_{C_1}(b_t) + \varphi^2_{C_1}(b_t) \le 1$. $\omega_{C_1}(b_t) = \sqrt{1 - \sigma^2_{C_1}(b_t) - \tau^2_{C_1}(b_t) - \varphi^2_{C_1}(b_t)}$ is the refusal degree for the element $b_t \in B$ in the set C_1.

In the next section, we suggest a novel SF knowledge metric along with its properties.

3. A NOVEL SPHERICAL FUZZY ENTROPY MEASURE

The intuitionistic fuzzy set notion is expanded upon by the SFS concept. An SFN C_1 is a quadruple $\left(\sigma_{C_1}, \tau_{C_1}, \varphi_{C_1}, \omega_{C_1}\right)$ such that $0 \le \sigma_{C_1}, \tau_{C_1}, \varphi_{C_1}, \omega_{C_1} \le 1$ and $\sigma^2_{C_1} + \tau^2_{C_1} + \varphi^2_{C_1} + \omega^2_{C_1} = 1$ is true. Entropy measurements should be greatest at one point, the same as probability measurements, when all member functions of the SFS are equal $\left(\sigma^2_{C_1} = \tau^2_{C_1} = \varphi^2_{C_1} = \omega^2_{C_1} = \frac{1}{4}\right)$, and should be 0 when C_1 is a crisp set. So, keeping these facts into consideration, we introduce the axiomatic definition of an SF entropy measure.

Definition 4 A function $E: SFS(B) \to [0,1]$ is called an SF entropy metric if

(i) $E(C_1) = 0$ if and only if C_1 is a crisp set.

(ii) $E(C_1) = 1$ for $\sigma^2_{C_1}(b_t) = \tau^2_{C_1}(b_t) = \varphi^2_{C_1}(b_t) = \frac{1}{4}\ \forall\, b_t \in B$.

(iii) $E(C_1) \le K(C_2)$ when C_1 is crisper than C_2 i.e., $\sigma^2_{C_1}(b_t) \le \sigma^2_{C_2}(b_t) \le \frac{1}{4}, \tau^2_{C_1}(b_t) \le \tau^2_{C_2}(b_t) \le \frac{1}{4},\ \varphi^2_{C_1}(b_t) \le \varphi^2_{C_2}(b_t) \le \frac{1}{4}$ or $\sigma^2_{C_1}(b_t) \ge \sigma^2_{C_2}(b_t) \ge \frac{1}{4}, \tau^2_{C_1}(b_t) \ge \tau^2_{C_2}(b_t) \ge \frac{1}{4},\ \varphi^2_{C_1}(b_t) \ge \varphi^2_{C_2}(b_t) \ge \frac{1}{4}\ \forall\, b_t \in B$.

(iv) $E(C_1) = E((C_1)^c)$, where c represents the complement.

Now, we offer a novel SF entropy measure as given below.

$$E_{QG}(C_1) = \frac{1}{p}\sum_{t=1}^{p} 1 - \left[\left(\sigma_{C_1}(b_t) - \frac{1}{2}\right)^2 + \left(\tau_{C_1}(b_t) - \frac{1}{2}\right)^2 + \left(\varphi_{C_1}(b_t) - \frac{1}{2}\right)^2 + \left(\omega_{C_1}(b_t) - \frac{1}{2}\right)^2\right]^{\alpha}, \alpha \ge 1. \ (1)$$

Theorem 1 *The function* E_{QG} *is an SF entropy metric.*

Proof We will establish that E_{QG} has the properties (i)-(iv) of Definition 4.

(i) Let $E_{QG}(C_1) = 0$, then

$$1 - \left[\left(\sigma_{C_1}(b_t) - \frac{1}{2}\right)^2 + \left(\tau_{C_1}(b_t) - \frac{1}{2}\right)^2 + \left(\varphi_{C_1}(b_t) - \frac{1}{2}\right)^2 + \left(\omega_{C_1}(b_t) - \frac{1}{2}\right)^2\right]^{\alpha} = 0,$$

$$\Rightarrow \left(\sigma_{C_1}(b_t) - \frac{1}{2}\right)^2 + \left(\tau_{C_1}(b_t) - \frac{1}{2}\right)^2 + \left(\varphi_{C_1}(b_t) - \frac{1}{2}\right)^2 + \left(\omega_{C_1}(b_t) - \frac{1}{2}\right)^2 = 1.$$

$$\Rightarrow \sigma_{C_1}(b_t) + \tau_{C_1}(b_t) + \varphi_{C_1}(b_t) + \omega_{C_1}(b_t) = 1.$$

This gives the following four possibilities.

(a) $\sigma_{C_1}(b_t) = 1, \tau_{C_1}(b_t) = \varphi_{C_1}(b_t) = \omega_{C_1}(b_t) = 0\ \forall\, t$.
(b) $\tau_{C_1}(b_t) = 1, \sigma_{C_1}(b_t) = \varphi_{C_1}(b_t) = \omega_{C_1}(b_t) = 0\ \forall\, t$.
(c) $\varphi_{C_1}(b_t) = 1, \sigma_{C_1}(b_t) = \tau_{C_1}(b_t) = \omega_{C_1}(b_t) = 0\ \forall\, t$.
(d) $\omega_{C_1}(b_t) = 1, \sigma_{C_1}(b_t) = \tau_{C_1}(b_t) = \varphi_{C_1}(b_t) = 0\ \forall\, t$.

All of these possibilities indicate that C_1 is a crisp set.

Conversely, suppose that C_1 is a crisp set, then we have the following four possibilities.

(a) $\sigma_{C_1}(b_t) = 1, \tau_{C_1}(b_t) = \varphi_{C_1}(b_t) = \omega_{C_1}(b_t) = 0\ \forall\, t$.
(b) $\tau_{C_1}(b_t) = 1, \sigma_{C_1}(b_t) = \varphi_{C_1}(b_t) = \omega_{C_1}(b_t) = 0\ \forall\, t$.
(c) $\varphi_{C_1}(b_t) = 1, \sigma_{C_1}(b_t) = \tau_{C_1}(b_t) = \omega_{C_1}(b_t) = 0\ \forall\, t$.
(d) $\omega_{C_1}(b_t) = 1, \sigma_{C_1}(b_t) = \tau_{C_1}(b_t) = \varphi_{C_1}(b_t) = 0\ \forall\, t$.

All of these possibilities lead us to $E_{QG}(C_1) = 0$.

(ii) Let $\sigma_{C_1}(b_t) = \tau_{C_1}(b_t) = \varphi_{C_1}(b_t) = \frac{1}{2}\ \forall\, t$, then we have $\omega_{C_1}(b_t) = \frac{1}{2}$. So $E_{QG}(C_1) = 1$.

Conversely assume that $E_{QG}(C_1) = 1$. Then we have

$$\left(\sigma_{C_1}(b_t) - \frac{1}{2}\right)^2 + \left(\tau_{C_1}(b_t) - \frac{1}{2}\right)^2 + \left(\varphi_{C_1}(b_t) - \frac{1}{2}\right)^2 + \left(\omega_{C_1}(b_t) - \frac{1}{2}\right)^2 = 0.$$

$$\Rightarrow \left(\sigma_{C_1}(b_t) - \frac{1}{2}\right)^2 = 0, \left(\tau_{C_1}(b_t) - \frac{1}{2}\right)^2 = 0, \left(\varphi_{C_1}(b_t) - \frac{1}{2}\right)^2 = 0, \left(\omega_{C_1}(b_t) - \frac{1}{2}\right)^2 = 0\ \forall\, t.$$

$$\Rightarrow \sigma_{C_1}(b_t) = \frac{1}{2},\ \tau_{C_1}(b_t) = \frac{1}{2},\ \varphi_{C_1}(b_t) = \frac{1}{2}, \text{ and } \omega_{C_1}(b_t) = \frac{1}{2}\ \forall\, t.$$

(iii) Let C_1 be crisper than C_2 i.e., $\sigma_{C_1}^2(b_t) \le \sigma_{C_2}^2(b_t) \le \frac{1}{4}, \tau_{C_1}^2(b_t) \le \tau_{C_2}^2(b_t) \le \frac{1}{4}$, $\varphi_{C_1}^2(b_t) \le \varphi_{C_2}^2(b_t) \le \frac{1}{4}$ or $\sigma_{C_1}^2(b_t) \ge \sigma_{C_2}^2(b_t) \ge \frac{1}{4}, \tau_{C_1}^2(b_t) \ge \tau_{C_2}^2(b_t) \ge \frac{1}{4}$, $\varphi_{C_1}^2(b_t) \ge \varphi_{C_2}^2(b_t) \ge \frac{1}{4}$ $\forall\, b_t \in B$

Now, when $\sigma_{C_1}^2(b_t) \le \sigma_{C_2}^2(b_t) \le \frac{1}{4}, \tau_{C_1}^2(b_t) \le \tau_{C_2}^2(b_t) \le \frac{1}{4}$, $\varphi_{C_1}^2(b_t) \le \varphi_{C_2}^2(b_t) \le \frac{1}{4}$, then $\omega_{C_1}^2(b_t) \ge \omega_{C_2}^2(b_t) \ge \frac{1}{4}$.

Also, $\sigma_{C_1}(b_t) \le \sigma_{C_2}(b_t) \le \frac{1}{2}$, $\tau_{C_1}(b_t) \le \tau_{C_2}(b_t) \le \frac{1}{2}$, $\varphi_{C_1}(b_t) \le \varphi_{C_2}(b_t) \le \frac{1}{2}$.

$\Rightarrow \sigma_{C_1}(b_t) - \frac{1}{2} \le \sigma_{C_2}(b_t) - \frac{1}{2} \le 0, \tau_{C_1}(b_t) - \frac{1}{2} \le \tau_{C_2}(b_t) - \frac{1}{2} \le 0, \varphi_{C_1}(b_t) - \frac{1}{2} \le \varphi_{C_2}(b_t) - \frac{1}{2} \le 0.$

$\Rightarrow \left(\sigma_{C_1}(b_t) - \frac{1}{2}\right)^2 \ge \left(\sigma_{C_2}(b_t) - \frac{1}{2}\right)^2 \ge 0, \left(\tau_{C_1}(b_t) - \frac{1}{2}\right)^2 \ge \left(\tau_{C_2}(b_t) - \frac{1}{2}\right)^2 \ge 0, \left(\varphi_{C_1}(b_t) - \frac{1}{2}\right)^2 \ge \left(\varphi_{C_2}(b_t) - \frac{1}{2}\right)^2 \ge 0, \left(\omega_{C_1}(b_t) - \frac{1}{2}\right)^2 \ge \left(\omega_{C_2}(b_t) - \frac{1}{2}\right)^2 \ge 0.$

$$\Rightarrow \begin{Bmatrix} \left(\sigma_{C_1}(b_t) - \frac{1}{2}\right)^2 + \left(\tau_{C_1}(b_t) - \frac{1}{2}\right)^2 \\ + \left(\varphi_{C_1}(b_t) - \frac{1}{2}\right)^2 + \left(\omega_{C_1}(b_t) - \frac{1}{2}\right)^2 \end{Bmatrix}^{\alpha} \ge \begin{Bmatrix} \left(\sigma_{C_2}(b_t) - \frac{1}{2}\right)^2 + \left(\tau_{C_2}(b_t) - \frac{1}{2}\right)^2 \\ + \left(\varphi_{C_2}(b_t) - \frac{1}{2}\right)^2 + \left(\omega_{C_2}(b_t) - \frac{1}{2}\right)^2 \end{Bmatrix}^{\alpha}.$$

$\Rightarrow E_{QG}(C_1) \le E_{QG}(C_2)$.

(iv) $E_{QG}((C_1)^c) = E_{QG}(C_1)$ follows from the expression of $E_{QG}(C_1)$.

Hence $E_{QG}(C_1)$ is a measure of precision for SFSs.

The valuation characteristic of the proposed SF knowledge metric is now discussed.

Theorem 2 *For any* $C_1, C_2 \in SFS(B)$, *we have*

$$E_{QG}(C_1 \cup C_2) + E_{QG}(C_1 \cap C_2) = E_{QG}(C_1) + E_{QG}(C_2),$$

where $\cup$ *and* $\cap$ *denote respectively the union and intersection of the SFSs*.

4. COMPARATIVE ANALYSIS

Here, we compare how the proposed SF entropy metric performs against the existing information metrics through various aspects such as ambiguity computation, linguistic hedges, and attribute weight computation. We begin by listing the SF information measurements that are currently used in the literature.

Aydogdu and Gul [11]

$$E_{AG}(C_1) = \frac{1}{p}\sum_{t=1}^{p}\left(1 - \frac{4}{5}\left[\left|\sigma_{C_1}^2(b_t) - \tau_{C_1}^2(b_t) + \left|\omega_{C_1}^2(b_t) - 0.25\right|\right|\right]\right).$$

Barukab et al. [10]

$$E_{BAAAK}(C_1) = \frac{1}{2p}\sum_{t=1}^{p}\left(\left(1 - \left|\sigma_{C_1}^2(b_t) - \tau_{C_1}^2(b_t)\right|\right)\left(2 - \sigma_{C_1}^2(b_t) - \tau_{C_1}^2(b_t) - \varphi_{C_1}^2(b_t)\right)\right).$$

Li et al. [12]

$$E_{LLMY}(C_1) = \frac{1}{p}\sum_{t=1}^{p}\left(1 - \frac{1}{\sqrt{2}}\sqrt{\left(\sigma_{C_1}^2(b_t)\right)^2 + \left(\tau_{C_1}^2(b_t)\right)^2 + \left(\varphi_{C_1}^2(b_t)\right)^2 + \left(\omega_{C_1}^2(b_t) - 1\right)^2}\right).$$

4.1 Ambiguity Computation

Here, we use the suggested SF entropy metric for determining the imprecision content of SFSs and will contrast the results with the existing SF entropy metrics.

Example 1 Consider five SFSs $C_1, C_2, C_3, C_4,$ and C_5 in $B = \{b_1, b_2, b_3\}$ as shown below

$$C_1 = \{(b_1, 0.2, 0.3, 0.2), (b_2, 0.1, 0.5, 0.4), (b_3, 0.2, 0.8, 0)\},$$
$$C_2 = \{(b_1, 0.10, 0.48, 0.41), (b_2, 0.2, 0.3, 0.5), (b_3, 0.4, 0.1, 0.1)\},$$
$$C_3 = \{(b_1, 0.5, 0.3, 0.1), (b_2, 0.4, 0.2, 0.2), (b_3, 0.3, 0.5, 0)\},$$
$$C_4 = \{(b_1, 0.4, 0, 0.4), (b_2, 0.5, 0.4, 0.1), (b_3, 0.4, 0.21, 0.04)\},$$
$$C_5 = \{(b_1, 0.2, 0.2, 0.3), (b_2, 0.3, 0.1, 0.1), (b_3, 0.4, 0.4, 0)\}.$$

The ambiguous content of these five SFSs is given in Table 1.

Table 1 Ambiguity content of different SFSs concerning Example 1

	C_1	C_2	C_3	C_4	C_5
$E_{AG}(C_t)$	**-0.0834**	0.0461	0.0331	0.0545	**-0.0062**
$E_{BAAAK}(C_t)$	0.5102	**0.6315**	**0.6315**	0.6477	0.8000
$E_{LLMY}(C_t)$	0.4911	0.5833	**0.5958**	**0.5958**	0.7315
$E_{QG}(C_t)$	0.9265	0.9520	0.9623	0.9467	0.8650

($\alpha = 2.5$ in E_{QG}. Bold values denote irrational results).

Table 1 provides the following observations:

(1) The SF entropy function E_{AG} gives the ambiguous content of the two SFSs C_1 and C_5 to be negative, which is not rational.

(2) The SF entropy measure E_{BAAAK} gives the ambiguous content of two different SFSs C_2 and C_3 to be the same i.e. 0.6315, which is not satisfactory.

(3) The SF entropy measure E_{LLMY} gives the ambiguous content of two different SFSs C_3 and C_4 to be the same i.e. 0.5958, which is unreasonable.

(4) The suggested SF entropy measure E_{QG} computes the ambiguity of all five SFSs without any counterintuitive results.

4.2 Linguistic Hedges

Here, we give an example to show the behavior of the suggested SF entropy measure. We offer an example incorporating linguistic hedges to make it mathematically sound and practically acceptable. By using a linguistic example, we will choose the best information measure in the SF environment by using several linguistic variables such as “LARGE”, “quite LARGE”, “very LARGE”, “quite very LARGE”, “very very LARGE”, etc. First, we recall the definition of the modifier C_1^δ of an SFS C_1.

Definition 5 [5] For any $C_1 = \left\{\left(b_t, \sigma_{C_1}(b_t), \tau_{C_1}(b_t), \varphi_{C_1}(b_t)\right); b_t \in B\right\} \in SFS(B), C_1^\delta, \delta > 0$ is defined as

$$C_1^\delta = \left\{\left(\begin{array}{c} b_t, \left(\sigma_{C_1}(b_t) + \varphi_{C_1}(b_t)\right)^\delta - \left(\varphi_{C_1}(b_t)\right)^\delta, \\ \sqrt{1 - \left(1 - \left(\tau_{C_1}(b_t)\right)^2\right)^\delta}, \left(\varphi_{C_1}(b_t)\right)^\delta \end{array}\right); b_t \in B\right\}.$$

In order to analyze and compare the suggested SF entropy, we provide an example using structured linguistic data.

Example 2 Consider an SFS $C_1 \in SFS(B), B = \{b_1, b_2, b_3, b_4, b_5\}$ given as

$C_1 = \{(b_1, 0, 0, 0.1), (b_2, 0.1, 0.5, 0), (b_3, 0.2, 0.1, 0), (b_4, 0.1, 0.5, 0.2), (b_5, 0, 0, 0)\}.$

With the help of Definition 5, we define the SFSs as More or less LARGE $= C_1^{\frac{1}{2}}$, LARGE

$= C_1$, quite LARGE $= C_1^{\frac{3}{2}}$, very LARGE $= C_1^2$, quite very LARGE $= C_1^{\frac{5}{2}}$, very very LARGE $= C_1^3$.

We compare our suggested SF entropy function with the existing SF entropy functions for estimating the ambiguity of these SFSs. Fig. 1 and Table 2 show the results.

Table 2 Ambiguity content of SFS regarding Example 3

	E_{AG}	E_{BAAAK}	E_{LLMY}	E_{QG}
$C_1^{\frac{1}{2}}$	-0.3634	0.7910	0.7599	0.6127
C_1	-0.3242	0.8134	0.8282	0.4787
$C_1^{\frac{3}{2}}$	-0.3162	0.7783	0.7993	0.4097
C_1^2	-0.3166	0.7403	0.7632	0.3763
$C_1^{\frac{5}{2}}$	-0.3088	0.7109	0.7320	0.3561
C_1^3	-0.2099	0.6878	0.7064	0.3402

($\alpha = 2$ in E_{QG})

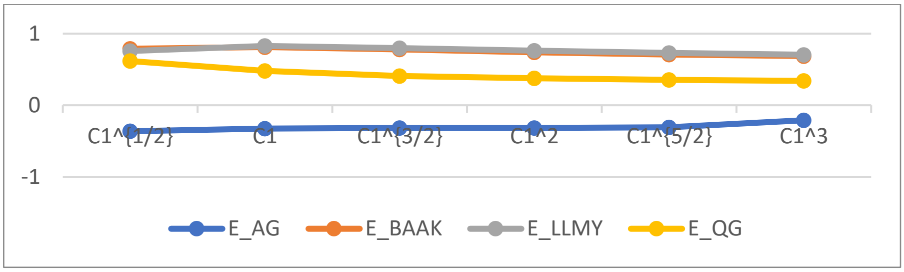

Figure 1 Behaviour of various SF entropy metrics concerning linguistic hedges

SF entropy metric E should satisfy the condition (2) because of the linguistic hedges characterization.

$$E\left(C_1^{\frac{1}{2}}\right) > E(C_1) > E\left(C_1^{\frac{3}{2}}\right) > E(C_1^2) > E\left(C_1^{\frac{5}{2}}\right) > E(C_1^3). \quad (1)$$

Thus, from Table 2 and Fig. 1, we conclude that the suggested SF entropy function is more reasonable in terms of linguistic variables because all of the existing SF entropy metrics do not exhibit the condition given in Eq. (1).

4.3 Attribute Weight Computation

In multi-attribute decision-making (MADM) problems, the computation of attribute weights is a big issue and the attribute weights have a key role in the selection of the best alternative. Here, we establish the utility of the suggested SF knowledge metric in the computation of weights of attributes and also compare the outcome with the available SF entropy metrics.

Example 3 Consider a MADM problem based on three alternatives $C_t, t = 1, 2, 3,$ and five attributes $D_s, s = 1, 2, 3, 4, 5$ in the form of an SF decision matrix as shown below.

$$E =$$

$$\begin{bmatrix} (0.14, 0.38, 0.20) & (0.2, 0.6, 0) & (0.6, 0.37, 0.17) & (0.42, 0.46, 0.45) & (0.2, 0.5, 0.5) \\ (0.18, 0.43, 0.3) & (0.4, 0.5, 0.2) & (0.5, 0.3, 0) & (0.66, 0.09, 0.16) & (0, 0.47, 0.2) \\ (0.4, 0.6, 0.2) & (0.3\ 0.3, 0.3) & (0.4, 0.1, 0.1) & (0.4, 0.3, 0.1) & (0.5, 0.4, 0.3) \end{bmatrix}$$

Now, we compute the weight of the attributes with the following entropy-based method.

$$w_s = W(D_s) = \frac{1-E(D_s)}{\sum_{s=1}^{5}(1-E(D_s))}, \quad s = 1, 2, 3, 4, 5. \tag{2}$$

Here E is an SF entropy measure. The attribute weights computed by utilizing the SF information metrics are presented in Table 3.

Table 3 Values of attribute weights concerning Example 3

	E_{AG}	E_{BAAAK}	E_{LLMY}	E_{QG}
w_1	0.2088	0.1883	**0.1835**	0.1789
w_2	**0.1921**	0.1880	0.1966	0.1553
w_3	0.2099	0.1931	**0.1835**	0.3055
w_4	0.1972	**0.2153**	0.2205	0.1872
w_5	**0.1921**	**0.2153**	0.2159	0.1730

(Bold values denote irrational results. $\alpha = 1.5$ in E_{QG}).

We see from Table 4 that the existing SF entropy metrics E_{AG}, E_{BAAAK} , and E_{LLMY} give the same weight to two distinct attributes, which is not reasonable. However, the suggested SF entropy metric E_{QG} gives proper attribute weights without any unreasonable results.

Next, we will introduce a new decision-making method i.e. COPRAS in the SF environment.

5. MULTICRITERIA DECISION-MAKING

Here we offer the COPRAS (Complex Proportional Assessment) method for SFSs and this is based on the novel SF knowledge function. Consider $Z = \{C_1, C_2, \dots, C_p\}$ to be the set of alternatives and $Y = \{D_1, D_2, \dots, D_q\}$ to be the set of attributes. We have to find out the most suitable alternative among all the alternatives $C_t, t = 1, 2, \dots, p$ by looking into the set of attributes. The alternative's information in accordance with attributes is given in the shape of SFNs in the decision matrix $E = [(\sigma_{ts}, \tau_{ts,} \varphi_{ts})]_{p\times q}$. The main steps of this method are as:

Step 1: Calculate the entropy of each attribute i.e., $E_{QG}(D_s), s = 1, 2, \dots, q$.

Step 2: Calculate the weight of each attribute $D_s, s = 1, 2, \dots, q$ by the following formula

$$w_s = W(D_s) = \frac{1-E_{QG}(D_s)}{\sum_{s=1}^{q}(1-E_{QG}(D_s))}, \quad s = 1, 2, \dots, q.$$

Step 3: Formulate the weighted decision matrix $F = [(\sigma'_{ts}, \tau'_{ts}, \varphi'_{ts})]_{p\times q}$, where $(\sigma'_{ts}, \tau'_{ts}, \varphi'_{ts}) = (w_s\sigma_{ts}, w_s\tau_{ts,} w_s\varphi_{ts})$.

Step 4: Compute the score function $U((\sigma'_{ts}, \tau'_{ts}, \varphi'_{ts}))$ for all $t = 1, 2, \dots, p$ and $s = 1, 2, \dots, q$ by using the following formula given by [8]

$$U((\sigma'_{ts}, \tau'_{ts}, \varphi'_{ts})) = \frac{1}{3}(2 + \sigma'_{ts} - \varphi'_{ts} - \tau'_{ts}).$$

Step 5: Calculate $V_t = \frac{1}{|BA|}\sum_{s\in BA} U\big((\sigma'_{ts}, \tau'_{ts}, \varphi'_{ts})\big)$ and $W_t = \frac{1}{|CA|}\sum_{s\in BA} U\big((\sigma'_{ts}, \tau'_{ts}, \varphi'_{ts})\big)$, where BA denotes the set of benefit attributes and CA indicates the set of cost attributes, for all $t = 1, 2, \ldots, p$.

Step 6: Calculate each alternative's relative weight $C_t, t = 1, 2, \ldots, p$ by using the following formula

$$T_t = V_t + \frac{\sum_{t=1}^{p} e^{V_t}}{e^{V_t}\sum_{t=1}^{p}\frac{1}{e^{V_t}}}.$$

Step 7: Calculate the priority order $R_t, t = 1, 2, \ldots, p$ by the following expression:

$$R_t = \frac{T_t}{\max T_t} \times 100.$$

Step 8: Rank the alternatives in the descending order of the values of priority order and the alternative with the highest priority order value is the most suitable alternative.

Now, we illustrate the SF COPRAS method with the help of a decision-making problem based on the selection of the best market segment.

5.1 Selection of Best Market Segment

Businesses have been concentrating on green growth and sustainable development strategies in order to minimize the negative effects of business on the environment and to safeguard it in recent years. Furthermore, the way that consumers make purchases has evolved. When it comes to eco-friendly and organic products, they are prepared to spend a premium. Managers should so focus on creating market sectors appropriately. We will use our model in this section to determine which market category is ideal.

Example 6 Let $C_1, C_2, C_3,$ and C_4 be the four market sectors. The assessment of these options is conducted according to six criteria (D_1) Development opportunities, (D_2) Market, (D_3) Customer satisfaction, (D_4) Market size, (D_5) Sales volume, (D_6) Profitability identification with the help of linguistic variables and is expressed in SFNs in the decision matrix E below.

$$E = \begin{bmatrix} (0.255, 0.745, 0.255) & (0.410, 0.490, 0.410) & (0.135, 0.865, 0.135) & (0.200, 0.335, 0.335) \\ (0.335, 0.665, 0.335) & (0.500, 0.500, 0.500) & (0.590, 0.410, 0.410) & (0.955, 0.045, 0.045) \\ (0.500, 0.500, 0.500) & (0.500, 0.500, 0.500) & (0.500, 0.500, 0.500) & (0.590, 0.410, 0.410) \\ (0.590, 0.410, 0.410) & (0.090, 0.110, 0.310) & (0.590, 0.410, 0.410) & (0.955, 0.045, 0.045) \\ (0.590, 0.410, 0.410) & (0.955, 0.045, 0.045) & (0.590, 0.410, 0.410) & (0.100, 0.110, 0.110) \\ (0.500, 0.500, 0.500) & (0.500, 0.500, 0.500) & (0.590, 0.410, 0.410) & (0.000, 0.000, 0.200) \end{bmatrix}$$

Using the offered entropy function E_{QG}, we compute the entropy of each attribute and obtain $E_{QG}(D_1) = 0.9206, E_{QG}(D_2) = 0.8375, E_{QG}(D_3) = 0.9962, E_{QG}(D_4) = 0.7772,$ $E_{QG}(D_5) = 0.7246,\ E_{QG}(D_6) = 0.8366.$

Next, we form the weighted decision matrix by using (Step 3) as shown below

$$F = \begin{bmatrix} (0.022, 0.065, 0.022) & (0.035, 0.042, 0.035) & (0.011, 0.175, 0.118) & (0.017, 0.029, 0.029) \\ (0.060, 0.119, 0.060) & (0.089, 0.089, 0.089) & (0.105, 0.073, 0.073) & (0.171, 0.008, 0.008) \\ (0.002, 0.002, 0.002) & (0.002, 0.002, 0.002) & (0.002, 0.002, 0.002) & (0.002, 0.001, 0.001) \\ (0.144, 0.100, 0.100) & (0.022, 0.027, 0.076) & (0.144, 0.100, 0.100) & (0.234, 0.011, 0.011) \\ (0.179, 0.124, 0.124) & (0.289, 0.013, 0.013) & (0.179, 0.124, 0.124) & (0.030, 0.033, 0.033) \\ (0.090, 0.090, 0.090) & (0.090, 0.090, 0.090) & (0.106, 0.073, 0.073) & (0.000, 0.000, 0.036) \end{bmatrix}$$

Now, we compute the scores of all the SFNs given in the weighted decision matrix F by using (Step 4) and these values are presented in Table 4.

Table 4 Scores of the SFNs

	D_1	D_2	D_3	D_4	D_5	D_6
C_1	0.6449	0.6270	0.6660	0.6478	0.6434	0.6366
C_2	0.6524	0.6368	0.6660	0.6397	0.7542	0.6366
C_3	0.6414	0.6529	0.6660	0.6478	0.6434	0.6529
C_4	0.6530	0.7183	0.6663	0.7375	0.6545	0.6547

Next, we compute V_t, W_t, relative weight T_t and R_t for all $t = 1, 2, 3, 4$ (Table 7). Finally, the ranking of four market sectors in decreasing order of the values of $R_t, t = 1, 2, 3, 4$ is presented in Table 5.

Table 5 Ranking of alternatives

	V_t	W_t	T_t	R_t	Ranking
C_1	0.6442	0.6449	3.9923	99.4972	3
C_2	0.6667	0.6524	3.9900	99.4395	4
C_3	0.6526	0.6414	4.0125	100	1
C_4	0.6863	0.6530	4.0077	98.8799	2

From Table 5, we arrive at the result that (C_3) is the most suitable and effective market segment.

We compare the outcomes of the offered decision-making technique with several available techniques as shown in Table 6.

Table 6 Ranking results by various available methods

Method	Ranking results
Spherical fuzzy AHP-TOPSIS [13]	$C_3 > C_4 > C_1 > C_2$
Interval-valued MCDM method [14]	$C_3 > C_4 > C_1 > C_2$
MACBETH [15]	$C_3 > C_4 > C_1 > C_2$
Preference selection index method [16]	$C_3 > C_4 > C_2 > C_1$
Combinatorial mathematics-based decision-making method [17]	$C_3 > C_4 > C_2 > C_1$
COPRAS method (This paper)	$C_3 > C_4 > C_1 > C_2$

We observe that the best market sector is (C_3) as shown by all the methods including our suggested one. This establishes the validity and effectiveness of the suggested COPRAS method.

5. CONCLUSIONS

Entropy measure in a fuzzy/non-standard fuzzy environment is very useful in the computation of ambiguity and mainly in attribute weight computation in a multi-attribute decision-making problem. A new set of fundamental conditions necessary for a function to be an SF entropy function has been given along with a novel SF entropy function. The suggested SF entropy metric has satisfied all the necessary axiomatic requirements. The limitations of all of the available SF entropy functions have been highlighted particularly in the areas concerning ambiguity computation, attribute weight computation, and linguistic hedges. These limitations have all been addressed by the suggested entropy function without any counterintuitive results. A multi-attribute decision-making technique known as COPRAS in the SF area has been proposed with the aid of suggested entropy measure. The application of the developed COPRAS technique has been established in the selection of a suitable market sector. Also, the suggested method has promising future studies on different fuzzy sets.

REFERENCES

1. L. A. Zadeh, "Fuzzy sets", *Information and Control*, Vol. 8, 1965, pp. 338–353.
2. K. T. Atanassov, "Intuitionistic fuzzy sets", *Fuzzy Sets and Systems*, Vol. 20, 1986, pp. 87–96
3. R. R. Yager, "Pythagorean fuzzy subsets", in *2013 Joint IFSA World Congress and NAFIPS Annual Meeting IEEE*, 2013, pp. 57–61
4. B. C. Cuong and V. Kreinovich, "Picture fuzzy sets - A new concept for computational intelligence problems" in *2013 Third World Congress on Information and Communication Technologies (WICT 2013) IEEE*, 2013, pp. 1–6
5. T. Mahmood, K. Ullah, Q. Khan, N. Jan, "An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets", *Neural Computing and Applications*, Vol. 31, 2019, pp. 7041–7053
6. S. Ashraf, S. Abdullah, T. Mahmood, F. Ghani, and T. Mahmood, "Spherical fuzzy sets and their applications in multi-attribute decision making problems", *Journal of Intelligent and Fuzzy Systems*, Vol. 36, 2019, pp. 2829–2844
7. F. Kutlu Gundogdu and C. Kahraman, "Spherical fuzzy sets and spherical fuzzy TOPSIS method", *Journal of Intelligent and Fuzzy Systems*, Vol. 36, 2019, pp. 337–352
8. S. Ashraf and S. Abdullah, "Spherical aggregation operators and their application in multiattribute group decision-making", *International Journal of Intelligent Systems*, Vol. 34, 2019, pp. 493–523
9. C. E. Shannon, "A mathematical theory of communication", The Bell System Technic Journal, Vol. 27, 1948, pp. 379–423
10. O. Barukab, S. Abdullah, S. Ashraf, M. Arif, and S. A. Khan, "A new approach to fuzzy TOPSIS method based on entropy measure under spherical fuzzy information", *Entropy*, Vol. 21, 2019, pp. 1231
11. A. Aydogdu and S. Gul, "A novel entropy proposition for spherical fuzzy sets and its application in multiple attribute decision-making", *International Journal of*

Intelligent Systems, Vol. 35, 2020, pp. 1354–1374
12. Z. Li, A. Liu, J. Miao, and Y. Yang, "A three-phase method for spherical fuzzy environment and application to community epidemic prevention management", *Expert Systems with Applications*, Vol. 211, 2023, pp. 118601
13. M. Mathew, R. K. Chakrabortty, M. J. Ryan, "A novel approach integrating AHP and TOPSIS under spherical fuzzy sets for advanced manufacturing system selection", *Engineering Applications of Artificial Intelligence*, Vol. 96, 2020, pp. 103988
14. M. Mathew and J. Thomas, "Interval valued multi criteria decision making methods for the selection of flexible manufacturing system", *International Journal Data and Network Science*, Vol. 3, 2019, pp. 349–358
15. P. Karande and S. Chakraborty, "Evaluation and selection of flexible manufacturing systems using MACBETH method", *International Journal of Services and Operations Management*, Vol. 16, 2013, pp. 123-144
16. K. D. Maniya and M. G. Bhatt, "The selection of flexible manufacturing system using preference selection index method", *International Journal of Industrial and Systems Engineering*, Vol. 9, 2011, pp. 330-349
17. R. V. Rao and M. Parnichkun, "Flexible manufacturing system selection using a combinatorial mathematics-based decision-making method", *International Journal of Production Research*, Vol. 47, 2009, pp. 6981–6998

Abdul Haseeb Ganie is currently working as a Post-doctoral Fellow at the Department of Mathematics, National Institute of Technology, Warangal-506004, Telangana, India. He received an M.Sc. (Mathematics) degree from the Central University of Kashmir and a PhD (Mathematics) from SMVD University, Katra-182320, Jammu and Kashmir, India.

Samirah Alzahrani is an Associate professor at the department of mathematics in Taif University in KSA since 2021. She received the Master degree in Topology from King Abdulaziz University (KAU), Jeddah, Saudi Arabia, in 2008, and the Ph.D. in Topology from KAU, in 2016.

Amira M Seif is an Assistant Professor at Future High Institute of Engineering in Fayoum, Egypt. She got her PhD from Cairo University.

Mohammad M. Khalaf is a professor from supreme council of Egyptian universities and also the Head of Department of Mathematics at Mustaqbal University Saudi Arabia. He got his PhD from Alazhar University.